HOW DEEP IS YOUR PLAYBOOK?

TRICIA MULDOON BROWN AND ERIC B. KAHN

Abstract. An American football season lasts a mere sixteen games played over seventeen weeks. This compact season leads to, fairly or unfairly, intense scrutiny of every player's performance and each coach's decision. Our goal in this paper is to determine a measure of complexity for the decision of choosing a defensive alignment on any given down. There are only four standard defensive formations, defined generally by the personnel on the field, but how a coach physically situates the players on the field can emphasize widely different defensive strengths and weaknesses. To describe the number of ways a coach achieves this goal, we utilize the notion of equivalence classes from abstract algebra to define classifications of defensive formations. Enumerative combinatorics is then necessary to count the number of fundamentally different defensive alignments through the application of binomial coefficients. Descriptions of the rules for the game and diagrams of different defensive alignments make this paper accessible to even the novice, or non-fan.

1. Introduction

Many people have fond memories of participating in team sports in elementary school, high school, and even college. We recall the thrill of a close game and the joy or sadness after a win or loss. As we age, less of our time is devoted to playing sports as other responsibilities take priority and our bodies simply are not capable of competing at the same level. Our role in sport transforms from being an active participant to being a spectator. Many of us still go to the gym, play a game of racquetball, or pick up the old golf clubs on a regular basis, but rarely do these friendly competitions end in the glory of a plastic trophy so coveted in our past years. Now as we focus our attention to watching our favorite professional, college, or club teams, our competitive edge discovers a different outlet: critiquing athletes and coaches who are still involved in the games we love. We consider one such game.

One cold Sunday during a snowy December, residents of Pennsylvania and Massachusetts settled down in front of their televisions to watch a classic sport, American football. The Pittsburgh Steelers are hosting one of their chief rivals, the New England Patriots, at Heinz field and the game is almost over. It has been a close game, and the Steelers have the ball with time for one last play. The populations of two states are waiting anxiously to see how the two teams will line up, when one insightful and mathematically inclined fan decides to try to calculate the magnitude of complexity of Patriots coach Bill Belichick's decision to pick a defensive alignment. Before she can begin her calculations and before we can get back to the final play of the game, we describe the background needed to understand this question.

2. The Game of Football and Mathematics

The game of football is a competition between two teams on a playing field that is 100 yards long and 53.3 yards wide. In the National Football League, each team is composed of 53 players of whom exactly 11 are on the field at any given time. The goal of the team on offense is to traverse the length of the field and reach the far end; the goal of the defense is to prevent their opponents from completing this task. To fix our orientation in space, we will view the field as being longer in the vertical direction and shorter in the horizontal. Defensive formations will always align facing down the field. The right and left sides of a formation will be with respect to the orientation of the reader.

There are typically four different positions or types of players in a defensive formation: linemen, linebackers, safeties, and cornerbacks. We will assume there is no distinction between different players of the same type. Different combinations of players on the field at one time determine a formation. The four basic formations used by a defense during a football game are: the 3-4 which is made up of 3 lineman and 4 linebackers, the 4-3 which is made up of 4 lineman and 3 linebackers, the nickel which is made up of 4 linemen, 2 linebackers, 3 corners, and the dime which is made up of 4 cornerbacks, 2 safeties, and at least one linebacker. Diagrams displaying the standard alignment of each formation are in Figures 1–4.

To organize vertically how far from the line of scrimmage players stand, the field is divided into tiers or levels stretching its entire width. The levels from front to back are called: defensive line, mid, nickel, and deep. To organize the field horizontally, we consider the middle 23.5 yards to be the interior of the field and the 15 yards on either side of the interior to be the exterior dividing the field into three sections. Restrictions determining where different types of players may align will be determined after we discuss the necessary mathematics. To answer our question we make use of two branches of mathematics, enumerative combinatorics and abstract algebra.

One of the goals of enumerative combinatorics is to count sets of objects satisfying certain conditions. That is, we consider the number of ways a given pattern or formation can be created. We will need to count *combinations*, unordered subsets of a given set. The number of ways to choose a combination of k elements from a set of n distinct elements is given by the binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

One of the goals of abstract algebra is to describe formally the structure and relationships between objects in a set or between the sets themselves. The notion of an equivalence relation plays an important role in our problem. An *equivalence relation* $\mathcal{R}$ on a set S is a subset of all pairs in $S \times S$ that is reflexive, symmetric, transitive. A standard example of an equivalence relation on the set of real numbers $\mathbb{R}$ is $x\mathcal{R}y$ if $|x| = |y|$.

We will use a similar type of relation while describing defensive alignments. Two defensive alignments will be termed *mirror equivalent*, or simply *equivalent*, if at each level the order of the players from right to left is reversed. For this to be an equivalence relation we also require a defensive alignment to be mirror equivalent to itself. The goal is to count the number of non-equivalent defensive alignments, thus combining combinatorics and algebra to describe a game many Americans watch every Sunday during the fall and winter months. However, we want the defensive alignments that are counted to represent actual situations. The following is a list of restrictions we will use in creating different alignments for a defense. The list is by no means exhaustive but it allows for most defensive alignments found during a football game.

(1) Linemen can line up only on the defensive line level.
(2) There must be at least two linemen on the field.
(3) At most two linebackers can line up at the defensive line level.
(4) All other linebackers must line up at the mid level.
(5) Corners can line up at any level.
(6) Safeties can line up at the deep or mid level.
(7) There must be exactly two safeties on the field with at least one at the deep level.
(8) Only corners will line up in the exterior regions of the field outside the linebackers and safeties.
(9) Two linebackers cannot stand side by side on the defensive line level.

3. Counting the Formations

We will first determine which players are fixed, e.g. the linemen are always on the line and one safety is always deep. We then insert the remaining types of players into the formation first the linebackers, then the other safety, and then the corners. At each stage we consider whether a formation is *self-symmetric*, denoted by *ss*, that is, the alignment is mirror equivalent only to itself. When inserting a player into a self-symmetric formation, we count equivalent placements or reflections only once. Once an alignment is not self-symmetric we count both placements. This is because in a mirror equivalence the size of the equivalence class may only be one or two. If an alignment is not self-symmetric, it has already been paired with its reflection. It is a representative of an equivalence class of size two. Thus, each place a new player can be added creates a distinct formation. We place the players in the interior of the field in this fashion, keeping count of which alignments are self-symmetric. Corners are the only players who line up on the exterior sides of the fields. It will be left to count the number of ways the corners can line up with respect to the mirror equivalence and without. We multiply the former by the number of self-symmetric formations, that is, representatives of equivalence classes of size one, and multiply the latter by the number of formations which are not self-symmetric. We consider each of the four types of formations.

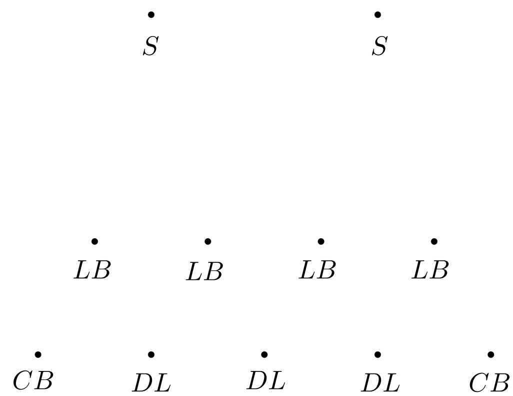

FIGURE 1. A standard 3-4 formation

The 3-4 Defense. The 3-4 defense is designed to be flexible enough to be effective against both running and passing plays. Figure 1 has an example of a 3-4 formation. Because of the restrictions, the three linemen, two linebackers, and one safety are fixed at specific levels. Next, there are three scenarios for placing the other two linebackers.

(1) Both on the line. This can be done in three ways:
   (a) both outside all linemen (ss), one way
   (b) one between two linemen and one outside all linemen in two ways, that is, either one or two linemen between the two linebackers on the line
   (c) both between two linemen (ss), one way.
(2) Both on mid level (ss) one way.
(3) One on the defensive line and one at the mid level. There are two spots for the one on the line:
   (a) inside in one way
   (b) outside in one way.

Now we place the remaining safety. For each formation the safety can be placed at the deep level in exactly one way. That is, we have three self-symmetric formations and four formations that are not self-symmetric where both safeties are deep. We determine the formations for the safety on the mid level.

(1) Parts (a) and (c) are self-symmetric, so by mirror symmetry, the mid level safety can be placed in each formation in two ways, between the two line backers (ss) and outside the line backers. Neither of the formations in part (b) is self-symmetric, so we count each position for the safety. He can stand in three different ways, on the outside on either side or in between the two line backers.
(2) This formation is self-symmetric. We have three ways to place the last safety; in the center of the mid level with two line backers on either side (ss), between two linebackers with one on one side and three on the other, and outside the linebackers.
(3) Neither part (a) nor part (b) is self-symmetric. Therefore, the safety can be placed on mid level in four ways, between the linebackers two ways and outside the linebackers two ways, for each formation.

So, we have $1 \cdot 2 + 1 \cdot 1 = 3$ self-symmetric formations with one safety on the mid level and $1 \cdot 2 + 3 \cdot 2 + 2 \cdot 1 + 4 \cdot 2 = 18$ formations where the safety is on the mid level and which are not self-symmetric. Including the cases where

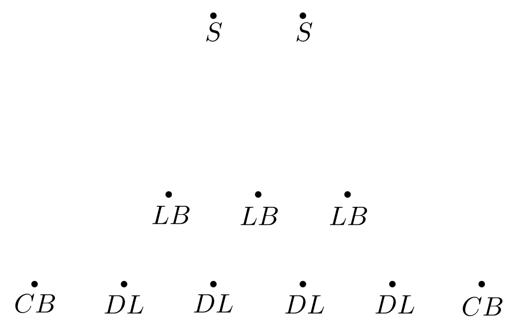

FIGURE 2. A standard 4-3 formation

both safeties are deep we have six self-symmetric formations and 22 alignments which are not self-symmetric. See figures 5, 6, and 7 for the progression of aligning the interior section of the 1b. alignment.

It is left to position the two corner backs. We have two cases.

(1) In the first, both corners stand on the same side and same level. This can be done in eight ways. Applying the mirror equivalence, each of the eight formations where the corners are standing on the same line and same side are symmetric to exactly one other. We have four different formations respecting mirror equivalence.
(2) In the second case, each corner stands on a different side or level. We have $\binom{8}{2} = 28$ possible places to line up the corners. With mirror symmetry there are four formations of the 28 which are self-symmetric, namely the ones where the corners line up on the same line. That leaves 12 placements that have been counted twice, that is, 4+12=16 total alignments with respect to mirror equivalence.

That gives 8+28=36 possible formations without regard to symmetry which we apply to the formations of linemen, linebackers, and safeties that are not self-symmetric. There are 4+16= 20 possible formations with regard to the mirror equivalence that we apply to the self self-symmetric formations. Thus we have

$$20 \cdot 6 + 36 \cdot 22 = 912$$

possible 3-4 defensive formations.

The 4-3 Defense. The 4-3 defense is common among NFL teams and is designed to defend against offensive running plays. The personnel on the field are always four linemen, three linebackers, two corners, and two safeties. See Figure 2. From our restrictions, the four linemen must line up on the defensive line level and one linebacker must line up at the mid level. The other two linebackers have the freedom to line up at the mid or defensive line levels.

(1) If both linebackers are placed at the mid level then there is only 1 possible formation (ss).
(2) If one linebacker is at the mid level and one is at the defensive line level there are 3 formations:
    (a) outside all linemen one way
    (b) in between the linemen with one on a side and three on the other side one way
    (c) in between the linemen with two on each side 1 way (ss).
(3) If both linebackers are at the defensive line level then there are three formations:
    (a) both outside all linemen one way (ss)
    (b) one outside all linemen and one in between two linemen three ways
    (c) both in between lineman with exactly two linemen between the linebackers one way (ss)
    (d) one with two linemen on either side and one with a single lineman on one side and three on the other one way.

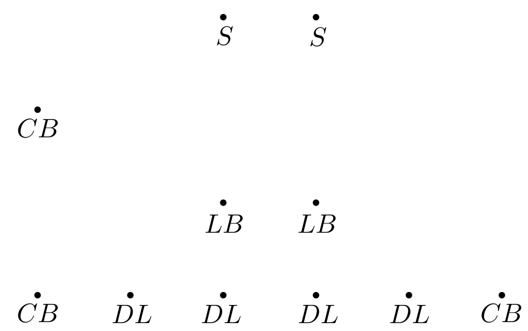

FIGURE 3. A standard Nickel formation

Next we place the remaining safety. In each case, the remaining safety may be placed at the deep level in exactly one way and this will respect the self symmetries, that is four self-symmetric formations and six that are not self-symmetric. Next we must count the number of formations with the safety at the mid level.

(1) The mid level safety can be placed outside all linebackers or in between two linebackers for a total of two ways, neither of which is self-symmetric.
(2) Parts (a) and (b) are not self-symmetric. The mid level safety can be placed in between the linebackers or outside the two linebackers in three ways resulting in six new formations which are not self-symmetric. Part (c) is self-symmetric, so the safety can be placed on the mid level in two ways, only one of which is still self-symmetric.
(3) Parts (a) and (c) are self-symmetric, so the mid level safety is placed on either side of the lone linebacker in one way for each formation. Parts (b) and (d) are not self-symmetric, hence the safety can stand on either side of the linebacker in two ways for each scenario. None of the new alignments are self-symmetric.

A defense will be self symmetric if and only if all levels of the defense are self symmetric meaning we have four self symmetric formations with both safeties deep and exactly one self symmetric formation with one safety at the mid level. For the non-self symmetric formations we have six formations with both safeties deep and with a safety at the mid level we have:

$$1 \cdot 2 + 2 \cdot 3 + 1 \cdot 1 + 2 \cdot 1 + 4 \cdot 2 = 19$$

formations. Thus we have five self symmetric formations and 25 non-self symmetric formations.

Lastly we must place the remaining two cornerbacks. In the 4-3 defense, there are the same number of cornerbacks as in the 3-4 defense and each has the same options of where to line up in the 3-4 defense. Thus, counting the alignments of cornerbacks in the 4-3 will be identical to that of the 3-4. So there are 36 possible alignments without regard to the mirror symmetry that we apply to the non-self symmetric formations of linemen, linebackers, and safeties and 20 possible formations that we apply to the self symmetric formations. This gives a total number of distinct defensive alignments in the 4-3 defense to be

$$36 \cdot 25 + 20 \cdot 5 = 1,000.$$

The Nickel Defense. The nickel defense is an altered version of the 4-3, primarily to protect against offensive passing plays. An example can be found in Figure 3. From our restrictions, the four linemen must line up on the defensive line level and both linebackers have the freedom to line up at the mid or defensive line levels. In terms of counting the alignments for linemen and linebackers, this is equivalent to the 4-3 formation as the extra linebacker in the 4-3 is fixed and thus has no effect on the counting problem. We omit the list of possible formations and instead will reference the linebacker possible alignments from the 4-3 formation while counting the rest of the nickel defense alignments.

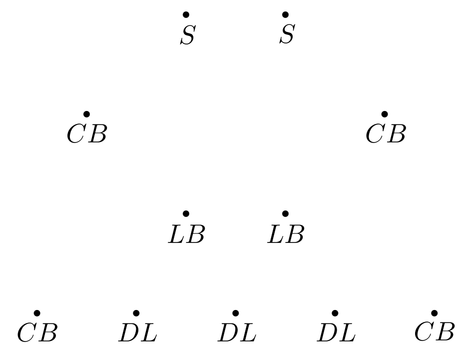

FIGURE 4. A standard Dime formation

Next we place the remaining safety. For each scenario, the remaining safety may be placed at the deep level in exactly one way and this will respect the self symmetries. Next we must count the number of formations with the safety at the mid level.

(1) The mid level safety can be placed outside all linebackers or in between two linebackers (ss) for a total of two ways.
(2) For parts (a) and (b) which are not self-symmetric the mid level safety can be placed on either side of the linebacker for a total of two ways. Because (c) is self-symmetric we apply mirror symmetry and place the safety in one way creating a new formation which is not self-symmetric.
(3) For all formations the mid level safety is placed as the lone player at the mid level one way. This gives two new self-symmetric formations and four new alignments which are not self-symmetric.

We recall a defense will be (ss) if and only if all levels of the defense are (ss), meaning that we have four self symmetric formations with both safeties deep and exactly $1 \cdot 1 + 1 \cdot 2 = 3$ self symmetric formation with one safety at the mid level. For the non-self symmetric formations we have six formations with both safeties deep and with a safety at the mid level we have $1 \cdot 1 + [2 \cdot 2 + 1 \cdot 1] + 4 \cdot 1 = 10$ formations. Thus we have seven self symmetric formations and 16 non-self symmetric formations.

Lastly we must place the remaining three cornerbacks. Since three cornerbacks will be on the field no alignment of them will be self symmetric and from our restrictions a cornerback can line up at any level of the defense giving each one four possible levels and we come to three scenarios.

(1) All three cornerbacks line up on the same side and same level of field. As with the above calculations, there are eight total possible alignments and four with respect to the mirror equivalence.
(2) Exactly two cornerbacks line up on the same side and level of the field. This gives eight possible regions for the pair of cornerbacks and seven for the lone cornerback yielding 56 total formations. If we take into account the mirror equivalence, each formation is equivalent to exactly one other so we have $56/2 = 28$ distinct alignments.
(3) All three cornerbacks line up in distinct regions of the field. This can be done in $\binom{8}{3} = 56$ ways without regard to the mirror equivalence. Each formation is mirror equivalent to exactly one other so we have $56/2 = 28$ distinct alignments.

Thus there is a total of $8 + 56 + 56 = 120$ formations and with respect to the mirror equivalence there are $4 + 28 + 28 = 60$ distinct formations. Thus we have $7 \cdot 60 + 16 \cdot 120 = 2,340$ ways to align the nickel defense.

The Dime Defense. The final defensive formation under consideration, the dime, is used exclusively against offensive passing plays. The dime formation has exactly four corners and two safeties with at least one linebacker. See Figure 4 for a sample formation. By the restrictions, one of the safeties is fixed at the deep level and there are at least two linemen on the field. We have the following cases:

(1) four linemen and one linebackers

(2) three linemen and two linebackers
(3) two linemen and three linebackers.

We place the linebackers on the defensive line or the mid level.

(1) (a) The linebacker stands on mid level (ss) one way.
    (b) The linebacker stands on defensive line level three ways; outside, inside with two linemen on either side (ss), and inside with one lineman on one side and three on the other side.
(2) (a) Both linebackers are on the mid level (ss) one way.
    (b) One linebacker is on the mid level and the other is on the defensive line level in two ways, inside or outside on the defensive line(neither is ss).
    (c) Both linebackers are on the defensive line level three way; both outside (ss), both inside (ss), and one inside and one outside in two ways, that is, with one lineman between the linebacker or two.
(3) (a) Three linebackers on the mid level (ss) one way
    (b) Two linebackers on the mid level and one on the defensive line level in two ways, inside (ss) in one way and outside in one way.
    (c) One linebacker on mid level and two on defensive line level in two ways, both outside (ss) and one inside and one outside.

We place the second safety. Again he can stand on the defensive line level or the mid level. If both safeties are deep we have eight self-symmetric formations and eight formations which are not self-symmetric. We consider the case where the second safety in on the mid level.

(1) (a) With symmetry the safety can stand on the mid level next to the linebacker one way.
    (b) The safety stands alone on the mid level one way forming a new self-symmetric formation and two others which are not.
(2) (a) The safety can be placed on mid level in two ways, between the two linebackers (ss) and outside the linebackers.
    (b) In either formation the safety can stand next to the linebackers in two ways, that is, to the left or right.
    (c) For all alignments the safety stands alone on the mid level in one way. We have two new self-symmetric formations and two new alignments which are not self-symmetric.
(3) (a) The safety can stand in two ways, inside the linebackers and outside the linebackers.
    (b) For the self-symmetric alignment, the safety can stand in two ways, between the linebackers (ss) and outside the linebackers. The other formation is not self-symmetric, so the safety can stand in three ways, that is, between the linebackers or on either side. We have created one self-symmetric formation and four new formations which are not self-symmetric.
    (c) Considering the formation which is self symmetric, the safety can stand one way outside the linebackers. The other formation is not self symmetric, so the safety can stand in two ways outside the linebackers, giving three new formations which are not self-symmetric.

Recall there are eight alignments that are self-symmetric and eight alignments that are not when both safeties are deep. Including these formations, to place the seven players in the middle of the field, we have

$$8 + 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 1 + 2 \cdot 2 + 1 \cdot 2 + 2 \cdot 1 + 1 \cdot 1 + 3 \cdot 1 + 1 \cdot 1 + 2 \cdot 1 = 27$$

formations that are not self-symmetric and

$$8 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 1 = 13$$

formations that are.

It is left to count the number of ways to place four cornerbacks. Since we allow a corner the freedom to line up on the same side and line of the field, we have five cases.

(1) All four cornerbacks stand on the same level on the same side of the field. There are four levels and two sides, so this can be accomplished eight ways without mirror symmetry. Otherwise each formation is equivalent to exactly one other, so under mirror equivalence we thus have four formations.

(2) Exactly three cornerbacks stand on the same level and on the same side of the field. Choose from eight regions to place the three standing together and seven to place the other cornerback. We have 56 formations without regard to mirror equivalence. Each formation is mirror symmetric to one other, so we have $8 \cdot 7/2 = 28$ formations.
(3) We form two groups of exactly two cornerbacks standing on the same level and same side of the field. This is equivalent to counting the placement of two cornerbacks standing on either a different level or a different side of the field. There are $\binom{8}{2} = 28$ ways to place these two groups. Four of these formations are self-symmetric and the rest are equivalent to exactly one other. We have 4+12=16 formations with regard to mirror symmetry.
(4) One group of two cornerbacks stands in the same region and the other two single cornerbacks stand in distinct regions. Place the set of two in eight ways. There are $\binom{7}{2} = 21$ ways to place the other two cornerbacks. Again each of these formations is mirror symmetric to exactly one other. We have $8 \cdot 21/2 = 84$ possible formations.
(5) All four cornerbacks stand in distinct regions of the field. This can be done in $\binom{8}{4} = 70$ ways. There are $\binom{4}{2} = 6$ that are self-symmetric. The other 64 have a reflection and are symmetric to exactly one other. We have 6+32=38 formations after applying mirror equivalence.

Thus there are

$$4 + 28 + 16 + 84 + 38 = 170$$

ways to place the four cornerbacks with mirror symmetry and

$$8 + 56 + 28 + 168 + 70 = 330$$

ways to place the cornerbacks without regard to mirror symmetry, so we have

$$170 \cdot 13 + 330 \cdot 27 = 11,120$$

total ways to build a dime defense.

4. Conclusion

After the calculations, the Patriots' fan bursts out that there are exactly 15,372 possible defensive alignments for Belichick to choose from. During her exclamation of how amazing it is for one coach to be able to choose exactly the correct defense in 30 seconds, the last play of the game begins to unfold. The Pittsburgh quarterback hikes the ball, scans the opposing defense, recognizes the formation, and decides the best plan of action. He sets his feet and heaves the ball towards the endzone. The populations wonder if Belichick's decision was the right one for their team.

Biographies

TRICIA MULDOON BROWN is an Assistant Professor of Mathematics at Armstrong Atlantic State University. Her research interests include algebraic and topological combinatorics, specifically studying the order complexes of certain poset products. After obtaining her doctorate at the University of Kentucky, and in the process becoming an avid fan of the Wildcats, Tricia and her husband moved south to new jobs in Savannah. When not teaching or researching, Tricia enjoys reading, walking her dogs, playing fantasy football, and cheering for the Steelers.

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ERIC BRENDAN KAHN is an Assistant Professor of Mathematics at Bloomsburg University. He wrote his dissertation on a topic from group theory that was motivated by algebraic topology while at the University of Kentucky. It was in Lexington that he developed his passion for teaching and decided to pursue his career at a teaching university. Aside from academia, Eric enjoys cooking, working out, reading, actively cheering for Boston's professional sports teams, and spending time with his wife and dog.

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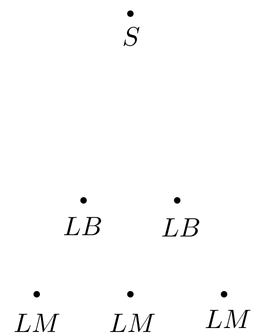

FIGURE 5. Fixed Personal in 3-4

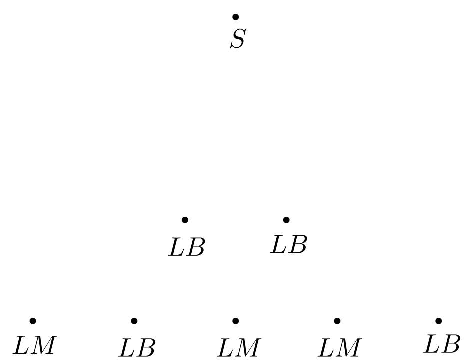

FIGURE 6. 3-4 Defense at Stage 1b. Alignment

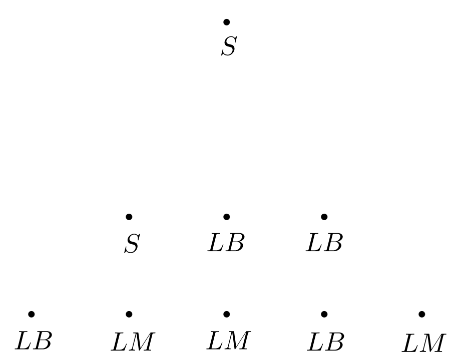

FIGURE 7. Interior 3-4 Defense in 1b. Alignment with the Safety at the Mid Level