How to Kick a Field Goal

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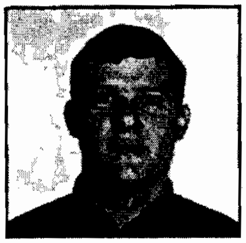

Daniel Isaksen is a graduate student in mathematics at the University of Chicago, studying algebraic topology. He became a college football fan while attending the University of California at Berkeley. Readers may recall his November 1995 *CMJ* article "Linear Algebra on the Gridiron," inspired by his participation in the Cal Marching Band.

Before a record crowd of 100,963, third-ranked California faced sixth-ranked Ohio State in the Rose Bowl on January 2, 1950. With only 1:55 remaining on the clock, the two teams were deadlocked in a 14–14 tie. Ohio State's drive for the goal had stalled at the California 6-yard line; it was now fourth down. Ohio State coach Wes Fesler decided to try for a field goal to take the lead. However, confusion with substitutes caused Ohio State to draw a 5-yard delay-of-game penalty. The field goal attempt was successful despite this seemingly foolish mistake, and Ohio State went on to win the game 17–14. According to the *New York Times* [**2**]:

> Fesler, in his post-game comment, said that he had deliberately sent in two substitutes just before the crucial field goal to draw a five-yard penalty and push the ball back from California's six-yard line to the eleven, "to give us a better angle."

Did Fesler make the right call? Is it really possible to improve the chances of scoring a field goal by backing up? I shall assume that the only factor affecting a kicker's chance of success is the angle subtended by the two goalposts at the kicking point. In particular, I shall ignore possible effects of the kicker's distance from the goalposts, such as the angle of elevation of the crossbar or the need to kick harder from a greater distance.

The Optimal Kicking Locus

Theorem. *Suppose the distance separating the two goalposts G and G' at one end of the field is $2g$. Let ℓ denote the perpendicular bisector of the line segment $G'G$, and $\ell(x)$ the line parallel to ℓ at a distance x. If $x < g$ then the closer a kicker on $\ell(x)$ is to the line of the goalposts, the larger is the angle subtended by the goalposts. But for $x \geq g$ the point K on $\ell(x)$ at which the goalposts subtend the maximum possible angle is characterized by the condition $\overline{KY} = \overline{YG}$, where Y is the closest point on ℓ to K, and G is the goalpost nearest $\ell(x)$.* (See Figure 1.)

Proof. For $x < g$ it is clear that the angle subtended by the goalposts increases monotonically as a kicker on $\ell(x)$ approaches the line $G'G$, reaching a maximum value of 180° when the kicker stands directly between the posts. For $x \geq g$, let K be the point on $\ell(x)$ where $\overline{KY} = \overline{YG}$. Thus the circle C with center Y and radius $\overline{YG}$ passes through both goalposts and is tangent to $\ell(x)$ at K, as in Figure 1. If K' is any point on $\ell(x)$ other than K, let L denote the point of intersection of C

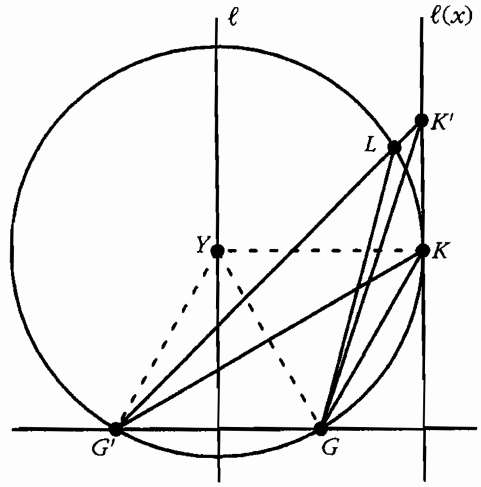

Figure 1. The angle $G'KG$ exceeds $\angle G'K'G$ by $\angle LGK'$.

and the line $K'G'$ joining K' with the farther goal G'. Then $\angle G'K'G < \angle G'LG$, since these two angles share a common side and K' is farther than L is from G'. But $\angle G'LG = \angle G'KG$, since these two angles inscribed in C subtend the same arc of this circle [**1**]. Thus the maximum angle subtended by the goalposts along $\ell(x)$ occurs at K. ■

Corollary. *In the Cartesian coordinate system with the line through the two goalposts as x-axis and the line ℓ as y-axis, the locus of points (x, y) at which the angle subtended by the goalposts is maximum is the portion of the hyperbola $x^2/g^2 - y^2/g^2 = 1$ above the x-axis.*

Proof. In the coordinate system specified, the coordinates of the points K, Y, and G in the theorem are $K = (x, y), Y = (0, y)$, and $G = (\pm g, 0)$, where the x-coordinate of G has the same sign as x. The condition $\overline{KY} = \overline{YG}$ characterizing K then becomes $\sqrt{(x-0)^2} = \sqrt{(0 \mp g)^2 + (y-0)^2}$, or $x^2 = g^2 + y^2$, which we recognize as the equation of the hyperbola $x^2/g^2 - y^2/g^2 = 1$, with vertices at G, G' and asymptotes $y = \pm x$ at 45° angles with the axes. ■

In the game of rugby, after a *try* (similar to a touchdown in American football) the scoring team attempts a kick for extra points. The ball must be kicked from directly behind the place where the try was scored; the kicker can choose the distance but no lateral movement is allowed. The corollary shows that the kicker's best choices lie on the hyperbola $x^2/g^2 - y^2/g^2 = 1$. In actual practice, kickers attempt to select points on the asymptotes instead, since it is relatively easy in the midst of a game to estimate these 45° angle lines through the middle of the goal.

Taking a Penalty to Improve the Kicking Angle

In American football the location of the ball at the beginning of a play is determined by the outcome of the previous play, but to allow running room on both sides the referee moves the ball laterally, if necessary, to place it between two *inbounds lines.*

These are lines parallel to the sidelines (see Figure 2), indicated on the playing field by "hash marks" on the yard lines. For field goal attempts the kicking field is therefore narrowed to the strip between the inbounds lines.

The question of whether to use a penalty to back up before attempting a field goal is complicated by the fact that penalties come in 5-yard increments. It is standard practice on field goal attempts to kick the ball from the point 7 yards directly behind where the referee placed the ball after the previous play. One could attempt a longer or shorter snap from center, kicking the ball from more or less than 7 yards behind the scrimmage line, and this would improve the angle on the goal if the resulting kicking point fell on the hyperbola, as in the rugby example discussed above. But the question that concerns us here is whether a 5-yard penalty, to move the kicking point back from 7 to 12 yards, would ever be advantageous.

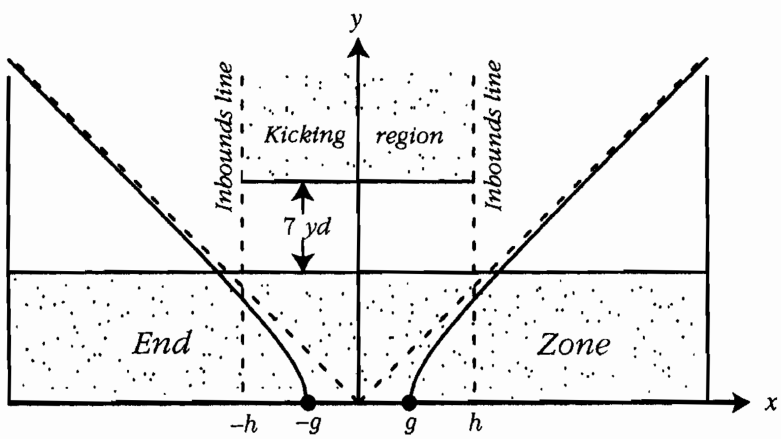

Figure 2. One end of a modern football field, showing the field goal kicking region bounded by inbounds lines and 7-yard line. Note the hyperbola $x^2/g^2 - y^2/g^2 = 1$ and its asymptotes.

Another complicating feature is that through the years the measurements of football fields have changed significantly, and even today the dimensions of the goal and the location of the inbounds lines differ in high school, collegiate, and professional football [3], [4]. Until 1974 the goalposts in professional football were at the front of the end zone, so the kicking point might have been as close as 7 yards from the line of the goalposts. In addition, the inbounds lines were much farther apart in professional football. But in today's game the goalposts are always at the back of the end zone, which is 10 yards deep, so the closest the kicker can be is 17 yards from the line of the goalposts.

Using the coordinate system described in the corollary, let (x, y) be the coordinates of the ball after the previous play, with $g \leq x \leq h$, where $2g$ is the distance between the goalposts and $2h$ is the distance between the inbounds lines. Then $K = (x, y+7)$ is the kicking point if no penalty is taken, and $K^* = (x, y+12)$ is the kicking point if a 5-yard penalty is taken before attempting the kick. Is $\theta^* = \angle G'K^*G$, the angle subtended by the goalposts at K^*, ever greater than $\theta = \angle G'KG$, the angle subtended by the goalposts at K?

From Figure 3 we see that

$$\theta^* = \tan^{-1}\left(\frac{x+g}{y+12}\right) - \tan^{-1}\left(\frac{x-g}{y+12}\right)$$

and

$$\theta = \tan^{-1}\left(\frac{x+g}{y+7}\right) - \tan^{-1}\left(\frac{x-g}{y+7}\right).$$

We use a computer algebra system to plot the level curves of the difference $a(x, y) = \theta^* - \theta$ for $g \leq x \leq h$ and $y \geq 0$, since it is advantageous to take a 5-yard penalty whenever $\theta^* > \theta$.

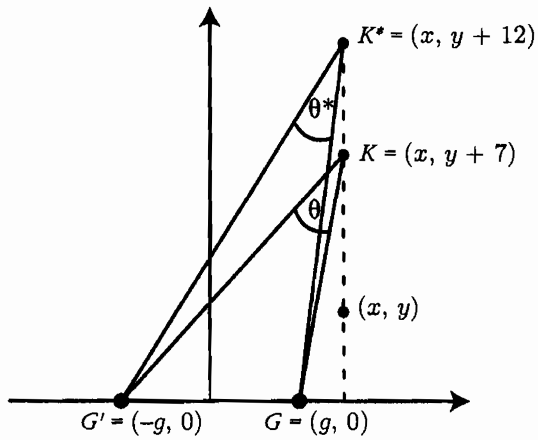

Figure 3. For what points (x, y) is $\theta^* > \theta$?

Figure 4 shows the level curves of $a(x, y)$, using the values of g and h in effect during the 1950 Rose Bowl game between Cal and Ohio State. At that time the collegiate regulation distance between the goalposts was 23 ft 4 in, and the distance between the inbounds lines was 53 ft 4 in. The top curve on each side in Figure 4 is $a(x, y) = 0$, so for (x, y) along this curve it is immaterial whether one takes the

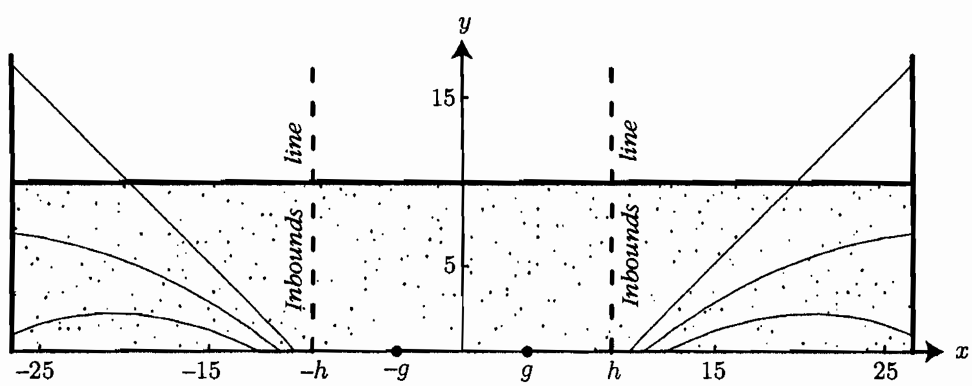

Figure 4. End zone, showing level curves $\theta^* - \theta = k\pi/180$, $k = 0, 1, 2$.

penalty or not before attempting the kick. At all points below this curve the angle subtended by the goalposts if the penalty is taken will be greater than without the penalty. The second highest curve is the level curve $a(x, y) = \pi/180$, so if the ball were on this curve before the start of a play, incurring a 5-yard penalty would give a 1° advantage. The remaining curve shows where the penalty would increase the kicking angle by 2°.

Note that the y-coordinate in Figure 4 gives the distance downfield from the goalposts. Thus if the goalposts were at the back of the end zone, as they were in the 1950 Rose Bowl contest, taking a 5-yard penalty would never give an advantage—the top curve in the figure lies outside the inbounds lines. Current strategy among football teams at all levels is not to seek such penalties; thus practice conforms with theory.

However, in professional football prior to 1974 the greater distance between the inbounds lines and the placement of the goalposts at the goal line created regions like those in Figure 4, in the corners near the goal line, where it was advantageous to back up 5 yards. And in fact professional teams did sometimes draw intentional penalties to improve their angle at the goal. But in the 1950 Rose Bowl, coach Fesler was mistaken if he believed the penalty gave his team a better angle.

Acknowledgment. Thanks to Alcibiades Petrofsky for suggesting this problem.

References

1. A. Coxford, D. Hirschhorn, and Z. Usiskin, *The University of Chicago School Mathematics Project: Geometry*, Scott, Foresman, Glenview, IL, 1991, p. 746.
2. Gladwin Hill, Ohio State field goal in last two minutes beats California in Rose Bowl, *New York Times*, January 3, 1950, p. 26.
3. National Collegiate Athletic Association, *1994 NCAA Football Rules and Interpretations*, NCAA, 1994.
4. National Federation of State High School Associations, *1994 High School Football Rules Book*, 1994.

Fascinating Sequences

To the Editor: Readers may be interested in a generalization of the ***minimal AP-free sequence*** discussed in the article "A Serendipitous Encounter with the Cantor Ternary Function," by L. F. Martins and I. W. Rodrigues, *CMJ* 27:3 (May 1996) 193–198. A problem on the 24th USA Mathematical Olympiad (April 1995) was this:

> Let p be an odd prime. The sequence $(a_n)_{n\geq 0}$ is defined as follows: $a_0 = 0, a_1 = 1, \ldots, a_{p-1} = p-2$; a_n is the least positive integer that does not form an arithmetic sequence of length p with any of the preceding terms. Prove that, for all n, a_n is the number obtained by writing n in base $p-1$ and reading the result in base p.

—Andrei Gnepp
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