Linear Algebra on the Gridiron

Daniel C. Isaksen

Daniel Isaksen received a B.A. in mathematics from the University of California, Berkeley. While studying there, he also played mellophone in the University of California Marching Band for four years. During that time, he served three years as assistant drum major for stuntsheets, three years as a rehearsal assistant, and two years as an equipment and facilities manager. Isaksen is currently a graduate student in mathematics at the University of Chicago.

Have you ever wondered how the intricate and precise marching patterns for a college football half-time show are created? Until just a few years ago, designing such shows was a time-consuming task requiring considerable technical skill. On a series of perhaps 30 pages, called "stuntsheets," the designer must record the band's formation on the football field at successive times in a show. Before computer software was developed to assist in making stuntsheets, labeled dots had to be placed by hand on a map of the field to show the locations of all band members at key moments. For a band of one hundred or more members this was a tedious process that required enormous care to make the transitions between successive stuntsheets uneventful. For example, one should avoid collisions—maneuvers that assign two band members to the same spot at the same time! Figure 1 shows a marching band formation on paper and on the gridiron.

As a student member of the marching band at the University of California, Berkeley, in 1989 Martin Haye realized that computers could not only dramatically ease the process of making stuntsheets, but also determine the marchers' locations at any time, thus producing an animation of the show. Commercial software applications were already available to assist show designers with the task of producing stuntsheets. They allowed the designer to arrange formations in a dynamic way, with movement corresponding to the intensity and flow of the music.

Unfortunately, Haye realized, these commercial programs were not well suited to the traditional marching style of the Cal Band. Most of the Cal Band's formations consist of rectangles, parallelograms, triangles, or straight lines, and many of the band's movements can be described by affine transformations (collinearity-preserving mappings, each being the composite of a linear transformation and a translation). For example, a line of band members might rotate 90° clockwise about its midpoint (linear transformation) and then move north 30 steps (translation).

Therefore, Haye started the ambitious project of writing his own program, and about a year later I joined him on the project. Armed with our knowledge of linear algebra, we set out to write a routine that would record the positions of the band members (an initial set of lattice points in the plane) under affine transformations. A plane affine transformation is determined by its effect on any three noncollinear points [2], so our plan was to specify the images of three noncollinear points under an affine transformation A, say $(x_1, y_1) \to (a_1, b_1)$, $(x_2, y_2) \to (a_2, b_2)$, $(x_3, y_3) \to (a_3, b_3)$, and then have the computer calculate the image of other points under A.

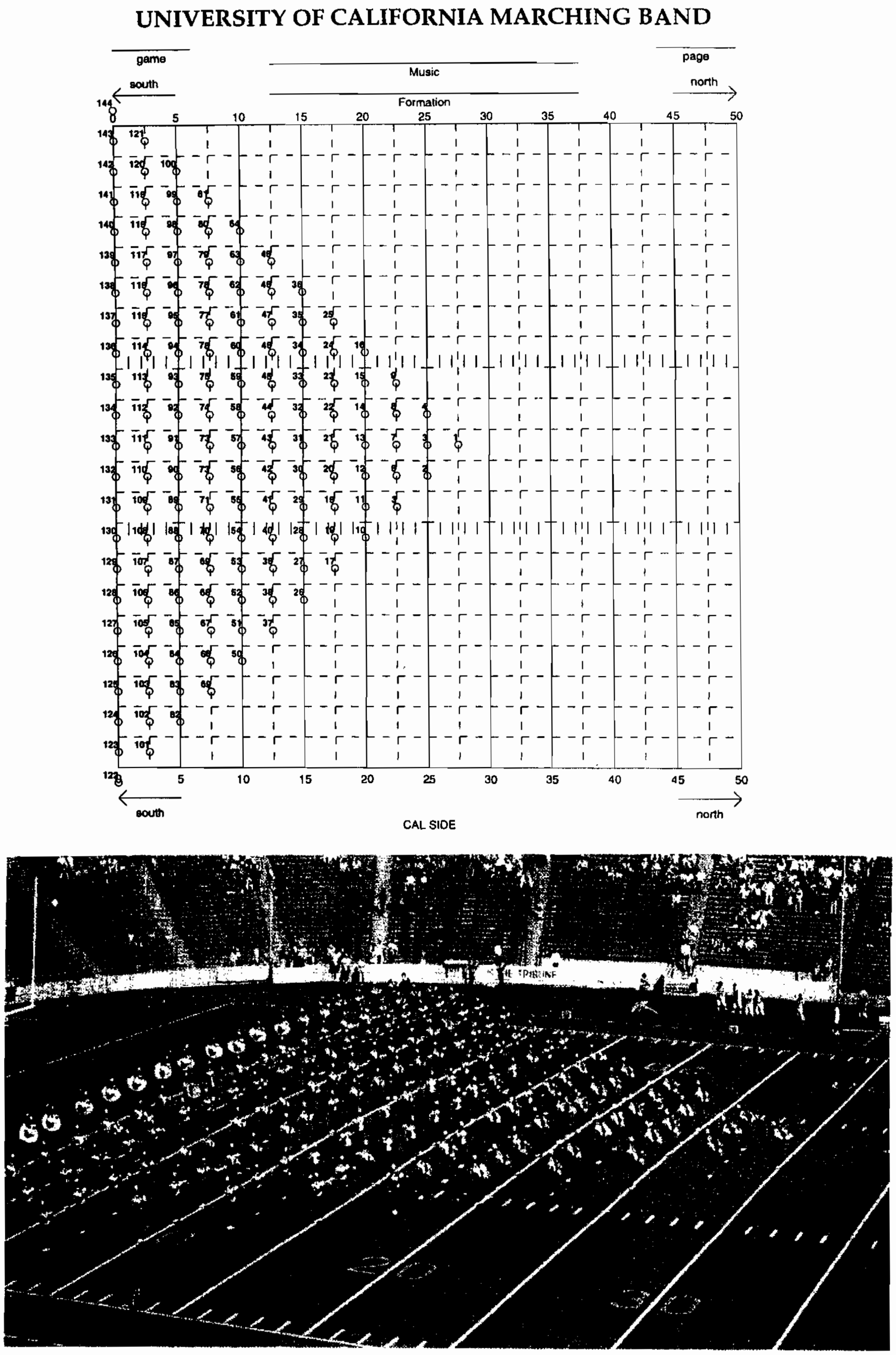

Figure 1
A Cal Band stuntsheet and corresponding formation.
(Photograph by Dan Cheatham. Reproduced from [1] with permission.)

If $A:(x,y) \to (rx+sy+t, ux+vy+w)$, then we wanted

$$\begin{bmatrix} r & s & t \\ u & v & w \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}. \tag{1}$$

The noncollinearity of the three given points (x_i, y_i) is equivalent to the linear independence of the vectors $(x_i, y_i, 1)$, or the invertibility of the matrix

$$B = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{bmatrix},$$

so the solution of the matrix equation (1) is

$$\begin{bmatrix} r & s & t \\ u & v & w \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} B^{-1}.$$

The inverse matrix can be calculated by Cramer's rule:

$$B^{-1} = \frac{1}{\det B} \begin{bmatrix} y_2 - y_3 & x_3 - x_2 & x_2 y_3 - x_3 y_2 \\ y_3 - y_1 & x_1 - x_3 & x_3 y_1 - x_1 y_3 \\ y_1 - y_2 & x_2 - x_1 & x_1 y_2 - x_2 y_1 \end{bmatrix},$$

where $\det B = x_1 y_2 - x_2 y_1 + x_3 y_1 - x_1 y_3 + x_2 y_3 - x_3 y_2$.

Upon testing our program, we found that it now took far less time to design a show. More important, we found significantly fewer transcription errors in the stuntsheets.

This successful application of linear algebra to simplify a familiar college activity tickled us. Next time you're enjoying the excitement of a marching band's half-time show at a college football game, remember the behind-the-scenes role that mathematics plays in the experience!

References

1. Cal Band History Committee, *The Pride of California: A Cal Band Centennial Celebration*, University of California, Berkeley, 1993.
2. H. S. M. Coxeter, *Introduction to Geometry*, 2nd ed., Wiley, New York, 1989, p. 203.