G.H. Hardy's Golfing Adventure

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A little-known paper by G.H. Hardy addresses a basic question in golf: which of two golfers of equal ability has the advantage, the more consistent golfer or the more erratic golfer? Hardy models golf as a sequence of independent shots, each of which can be normal, excellent or bad. In this paper, the distributions of hole scores for simulated golfers using Hardy's rules are computed. The distributions enable us to explore Hardy's basic question in different golfing contests.

There are very few sentences in print that contain both the word "golf" and the name G.H. Hardy. Hardy (1877-1947) was one of the most prolific and influential mathematicians of the early twentieth century. His book *A Mathematician's Apology* [1] makes the case for mathematics as a pure discipline of austere beauty and uncompromising standards. He wrote, "The mathematician's patterns, like those of the painter's or the poet's, must be beautiful, the ideas, like the colours or the words, must fit together in a harmonious way. There is no permanent place in the world for ugly mathematics." He found little beauty in applied mathematics. The assumptions made by applied mathematicians, tied to the laws of physics and other mundane concerns, are not always motivated by mathematical curiosity and the results are often more pragmatic than inspirational.

Given his background as a pure mathematician par excellence, his publication in the December 1945 issue of *The Mathematical Gazette* is singular. There, on pages 226 and 227, is an article titled "A Mathematical Theorem about Golf" by G.H. Hardy [2]. He introduces a simple model of golfing and provides a preliminary analysis. Here, we discuss Hardy's results and some calculations based on his model.

Hardy's Golf Problem

The problem proposed by G.H. Hardy is motivated by a hypothetical match between two golfers of equal ability. If one golfer is much more consistent than the other, which golfer has the advantage? Golfers are often faced with a choice of attempting a risky shot or playing safely. For instance, they may choose to try to hit over a lake directly at the hole, or play safely around the lake but farther from the hole. Is it better to use a cautious strategy or a risky strategy?

Hardy approached this question by imagining a golfer whose shots are either excellent (E), normal (N) or bad (B). The golfer plays a hole with a par (target score) of four. A player hitting four normal shots (NNNN) finishes with a score of 4. A bad shot adds one to the number of shots. Therefore, a player who hits three normal shots and then a bad shot (NNNB) has not finished the hole. Another normal shot (making the shot sequence NNNBN) finishes the hole with a score of 5. By contrast, an excellent shot reduces the required number of shots by one. Thus, a shot sequence of NNE finishes the hole with a score of 3. The sequence NBEN finishes the hole with a score of 4. More examples follow.

Shot Sequence	Score
NBNNN	5
BNNE	4
NEBBBN	6

A sequence can never end in B because a bad shot always adds one to the score. However, a sequence can end in E. The sequence ENN finishes the hole with a score of 3, as does the sequence ENE. In a sense, the golfer is cheated out of the benefit of an excellent shot, because ENN and ENE receive the same score. The first two shots (EN) leave the ball close to the hole, so a normal putt from this distance will go in. An excellent putt that goes into the exact center of the hole is enjoyable to watch, but the golfer gets no extra credit for perfection.

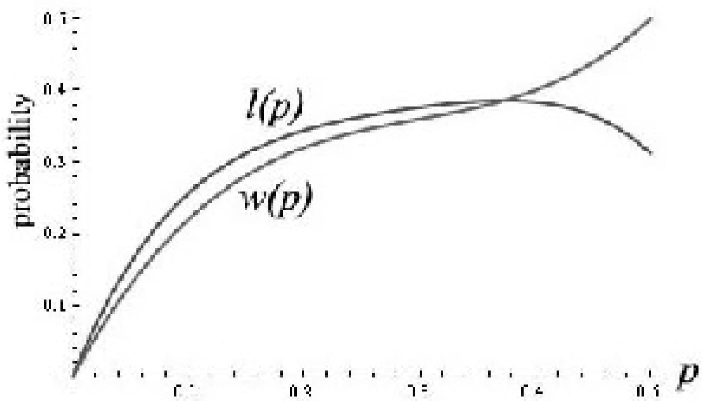
Our most important assumptions involve the distribution of shots. We suppose that all shots are independent, so a bad shot does not affect the probability that the next shot is excellent. (All golfers wish that this was realistic.) We assume that the probability of a bad shot is p with $0 \leq p \leq \frac{1}{2}$, and the probability of an excellent shot is the same, so the probability of a normal shot is $1 - 2p$. The only difference between one such golfer (the phrase "Hardy golfer" will refer to a golfer playing with these constraints) and another is the value for p. At first glance, all Hardy golfers appear to have equal ability, since excellent shots and bad shots have equal probability.

Suppose that golfer C is a Hardy golfer with p-value p_1 and golfer R is a Hardy golfer with p-value $p_2 > p_1$. Golfer C has a higher probability of hitting a normal shot than golfer R, so golfer C is more consistent (or more cautious golfer). Golfer R has a higher probability of hitting either an excellent shot or a bad shot than does golfer C, so golfer R is more erratic (or risky). The problem is to determine which is more likely to win a match.

Hardy's Analysis

Hardy's paper presents the case where $p_1 = 0$, so that golfer C always makes a par 4. Golfer R has a probability p of hitting an E shot (which Hardy calls a *supershot*) and probability p of hitting a B shot (which he calls a *subshot*). Hardy computes the probability of golfer R winning a hole as $w(p) = 3p - 9p^2 + 10p^3$. He then computes the probability that golfer R loses the hole as $l(p) = 4p - 18p^2 + 40p^3 - 35p^4$. Details are given in [4].

The graphs of $w(p)$ and $l(p)$ in Figure 1 show the probabilities of winning and losing.



Probabilities of a Hardy golfer with parameter p winning (w) and losing (l) a hole to a par 4.

For values of p less than approximately 0.37, player R is more likely to lose than win. The maximum vertical distance between the two curves occurs at approximately $p = 0.09$. Extensions to cases where the p-values for E and B shots are unequal are given in [3].

For realistic values of p, then, the consistent player is more likely to win the hole. Hardy says that this is at odds with the standard golfing wisdom that an erratic player is better off at match play (counting each hole as a separate contest) than at stroke play (where strokes are counted for all 18 holes). As we will see, the standard wisdom is actually correct, because the consistent player has an even larger advantage in stroke play.

Two Moments

To start to analyze a stroke play match between two Hardy golfers, we can calculate the mean score on a single hole. A reasonable guess is that the mean should be four, since E and B shots are equally likely. This would be correct if it were not the case that some E shots are wasted. In the sequence ENE, the second E does not improve the golfer's score, so the mean score for a Hardy golfer with $p > 0$ will be larger than 4.

We will calculate the mean and variance of the number of shots using a suggestion of Gregory Minton, by looking at the number of B shots.

If there are no B-shots, the golfer's score on the hole could be 2 (sequence EE) with probability p^2, 3 (sequences NNE, NEE or ENE) with probability $3p(1-2p)$ or 4 (sequence NNNN) with probability $(1-2p)^2$.

Generalizing, if there are k B-shots, the golfer's score could be $k+2$ if there are two E-shots also included. One of the E's must come at the end, so there are $k+1$ different sequences of this type. The probability of such a sequence is $(k+1)p^{k+2}$.

The golfer's score could be $k+3$ if there are two N-shots and an ending E. There are $\frac{1}{2}(k+2)(k+1)$ positions for the N-shots, so this type of sequence has probability $\frac{1}{2}(k+2)(k+1)p^{k+1}(1-2p)^2$. Further, the golfer's score could be $k+3$ if the sequence contains one E and one N before the last shot and ends in either N or E. This has probability $(k+2)(k+1)p^{k+1}(1-2p)(1-p)$.

The golfer's score could be $k+4$ if there are three N-shots before the end, with the last shot being either N or E. The probability is $\frac{1}{6}(k+3)(k+2)(k+1)p^k(1-2p)^3(1-p)$.

These are the only possibilities. It follows that the mean is

$$\begin{aligned} \mu &= \sum_{k=0}^{\infty}(k+2)(k+1)p^{k+2} + \sum_{k=0}^{\infty}(k+3)(k+2)(k+1)p^{k+1}(1-2p)(\tfrac{3}{2}-2p) \\ &\quad + \sum_{k=0}^{\infty}(k+4)\tfrac{1}{6}(k+3)(k+2)(k+1)p^k(1-2p)^3(1-p) \end{aligned}$$

and the second moment is

$$\begin{aligned} \mu_2' &= \sum_{k=0}^{\infty}(k+2)^2(k+1)p^{k+2} + \sum_{k=0}^{\infty}(k+3)^2(k+2)(k+1)p^{k+1}(1-2p)(\tfrac{3}{2}-2p) \\ &\quad + \sum_{k=0}^{\infty}(k+4)^2\tfrac{1}{6}(k+3)(k+2)(k+1)p^k(1-2p)^3(1-p) \end{aligned}$$

All of the sums can be evaluated using the geometric series $\displaystyle\sum_{k=0}^{\infty} p^k = \frac{1}{1-p}$ and differentiation. For example, multiplying by p^2 gives $\displaystyle\sum_{k=0}^{\infty} p^{k+2} = \frac{p^2}{1-p}$ and taking two derivatives produces $\displaystyle\sum_{k=0}^{\infty}(k+2)(k+1)p^k = \frac{2}{(1-p)^3}$. It follows that the first sum in the calculation of μ equals $\displaystyle\frac{2p^2}{(1-p)^3}$.

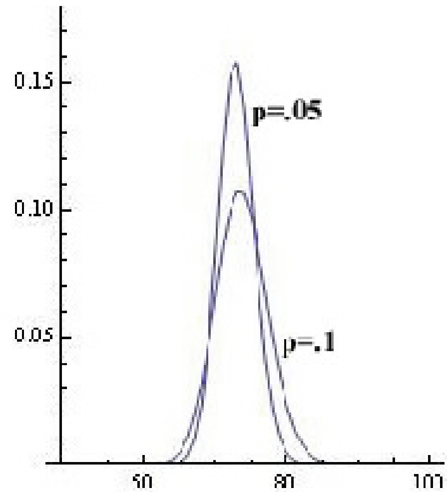
The mean and variance are

$$\begin{aligned} \mu &= 4 + p\left(1 - \frac{p^4}{(1-p)^4}\right) \\ \sigma^2 &= \mu_2' - \mu^2 = \frac{7p - 51p^2 + 156p^3 - 252p^4 + 219p^5 - 90p^6 + 10p^7}{(1-p)^8} \end{aligned}$$

For small values of p, the mean is approximately $4+p$ and the variance is approximately $7p+5p^2$. The approximations are quite good for $p < 0.2$. Higher p-values are unrealistic, as $p = 0.2$ implies that only 60% of the golfer's shots are normal.

What are realistic values for p? This is an awkward question, as it implicitly grants the model with more validity than it deserves. Clearly, golf shots come in more than three categories, and especially excellent shots or egregiously bad shots do not always modify the golfer's score by exactly one stroke. Nevertheless, to get an estimate of reasonable values I computed the variance of scores for one round of some PGA tournaments. They ranged between 8 and 12, corresponding to p-values between 0.06 and 0.09.

Graphs of the theoretical distributions of scores for 18 holes with $p = 0.05$ and $p = 0.1$ are shown below. All 18 holes are par 4's played by the rules described above.



The more erratic golfer with $p = 0.1$ has a wider distribution of likely scores, being more likely to shoot 69 or less and more likely to shoot 75 or higher. The mean with $p = 0.05$ is lower than the mean with $p = 0.1$. For 18 holes, they are approximately 72.9 and 73.8, respectively.

Stroke Play

Having a lower mean score does not necessarily imply that you will win a majority of your matches. The type of match being played influences who wins. Stroke play, tournaments and skins games are discussed here. Different forms of match play are discussed in [4].

In stroke play (also called medal play), each player counts all strokes over 18 holes and the lower total wins. The probabilities of players making different scores on a given hole can be combined to compute the probability of a player having a particular score for the entire round. Comparing these probabilities, a Hardy golfer with $p = 0.05$ will defeat a Hardy golfer with $p = 0.1$ 53% of the time, with 9% ties. This is illustrated below, where the results of 20 simulated rounds are shown.

p = 0.1: 76,72,76,74,68,78,72,68,71,77,75,76,76,74,71,73,83,80,70,70

p = .05: 76,74,75,73,71,73,73,74,69,70,71,75,71,74,73,75,72,74,73,74

The more erratic golfer ($p = 0.1$) records the best score (68, twice) and the eight worst scores (76-83). If the scores are of twenty stroke play contests, the more consistent golfer has the lower score 10 times (50%) and there are two ties (10%).

In stroke play, the consistent player has a notable advantage over the erratic player, winning 53% of the matches against only 38% losses. The advantage is reduced in match play, with 46% wins against 42% losses over an 18-hole match. To find a competition in which the risk-taking erratic player has an advantage, we turn to the ultimate game for aggressive players.

Skins Game

In a skins game, all players in a foursome record their scores on a hole and the lowest score wins. If two or more players tie with the lowest score, then the group moves on to the next hole. The phrase "two tie, all tie" means that no one is eliminated if there are at least two tied for low score. There are many ways of betting on skins. The most common is to carry over all bets. If the bet is $1 per hole and the first hole is tied by two or more players, then everybody plays the second hole for $2. If the second hole is also tied, then everybody plays the third hole for $3. An erratic golfer can have a string of bad holes and still be in line to win all the money if the other three players tie every hole.

Our first analysis of Hardy golfers playing skins will look at a foursome consisting of three C-golfers ($p = 0.05$) and one R-golfer ($p = 0.1$). The calculations show that each of the C-golfers has a 8.2% chance of winning a given hole, the R-golfer has a 15.1% chance of winning it, and 60.4% of the time there is a tie. The erratic golfer has an advantage here, although with several carry-overs one of the consistent players could win all of the skins.

If the foursome consists of two C-golfers and two R-golfers, the percentages change. Each of the C-golfers has a 7.7% chance of winning a hole, each of the R-golfers has a 13.8% chance and 43% of the time there is a tie. With two erratic golfers having a chance to break loose, less than half of the skins are carried over. The number of ties depends on the composition of the foursome. The chance of a tie increases to 55% if there are three R-golfers and one C-golfer. In this case, each R-golfer has a 12.6% chance of winning a hole while the C-golfer has a 7.4% chance of winning a hole.

In all cases, erratic golfers have an advantage over consistent golfers. The actual number of carry-overs in a real skins match depends on a number of factors that we have ignored. For example, some holes are designed to tempt golfers to take large risks, effectively increasing the difference in p-values. As more holes are tied, pressure can build and affect risks that a player is willing to take. Psychologically, it is great fun to make a tying putt that keeps an opponent from winning a skin, whereas having a putt to win several skins emboldens some and tightens up others. This may increase the number of tied holes.

Tournament Golf

The other main setting in which golf is played is a tournament. The most basic tournament has everybody play one round, with the lowest score winning. Professionals typically play four rounds. Amateurs often play in tournaments in which scores are adjusted based on handicaps. Each of these cases will be considered.

Suppose we have a one round tournament contested by 70 consistent Hardy golfers with $p = 0.05$ and 70 erratic Hardy golfers with $p = 0.1$. An interesting statistical paradox is present. We have seen that the consistent golfers have a lower mean score than the erratic golfers, nearly a full stroke better on the average, 72.9 to 73.8. However, in the tournament there is a 71% chance that one of the erratic golfers will have the lowest score.

The golfers with the highest average scores are extremely likely to produce the single lowest score! The paradox is resolved by noting that the scores of the erratic golfers have a higher variance, giving the erratic golfers a higher probability of a better score. This was seen in the simulation of 20 scores each for consistent and erratic golfers. A particular erratic golfer is not a good bet to score well, but the odds are good that at least one of them will go low and have an excellent score.

The odds shift when the tournament has more rounds. An erratic golfer who gets lucky the first round is subject to the same distribution of scores for the second round, a distribution that is centered around a not-so-stellar 73.8. Over four rounds, the erratic golfers retain a slight edge, winning 60% of the tournaments. The longer the tournament continues, the more likely it is that the steady, unspectacular golfers will prevail.

Handicaps

An important feature of the golf handicap system is revealed by the Hardy golfing model. The goal of a handicap system is to even the odds in a competition between unequal golfers. Golfer A who averages 90 could play golfer B who averages 70 if a handicap of 20 strokes were given. Then a better-than-average 87 (net 67) by golfer A would beat a worse-than-average 74 by golfer B.

The details of the handicap system administered by the United States Golf Association (USGA) are complicated (and surprisingly mathematical). Each course is evaluated and assigned a course rating and a slope rating. These determine a line $y = mx + b$ that predicts what a golfer of a given handicap would be expected to shoot on that course. A slope rating of 113 is average. On a course with course rating 70.5 and slope rating 130, we have $m = \frac{130}{113} \approx 1.15$ and $b = 70.5$, so that a golfer with handicap 10 should average $1.15(10) + 70.5 = 82$.

To compute a handicap, only the best ten of the last twenty scores (relative to the course rating and slope rating) count. The USGA's explanation acknowledges that the system tries to estimate a player's *potential* and not the typical performance.

To illustrate the difference, let us look at the simulated scores shown in the "Stroke Play" section. Assume that each round is played on a course with course rating 70 and slope rating 113. The ten best scores for each golfer are retained and then 70 is subtracted.

$$p = 0.1\text{: } 72,68,72,68,71,74,71,73,70,70 \rightarrow 2,-2,2,-2,1,4,1,3,0,0$$
$$p = .05\text{: } 73,71,73,73,69,70,71,71,73,72 \rightarrow 3,1,3,3,-1,0,1,1,3,2$$

The averages of the adjusted scores are 0.9 for the ($p = 0.1$) R-golfer and 1.6 for the ($p = 0.05$) C-golfer. Rounding off, the handicaps would be 1 and 2, respectively. In a match between the players, the C-golfer would get a stroke, meaning that the C-golfer's score would be reduced by 1 before comparing to the R-golfer's score. In the 20 simulated matches, the C-golfer won 10 and tied 2. Subtracting a stroke unbalances the match further, with the C-golfer winning 12 and tying 1.

This shows that in a head-to-head match the USGA handicap system favors consistent players. An erratic player is rated on the potential indicated by the ten best rounds, which can be better than average if the player's scores have a high variance. Since better players tend to be more consistent, the USGA handicap system tends to favor better players.

If you think this is unfair, consider our tournament analysis. The higher-average, erratic players dominated. Out of a large group of erratic players, it is likely that one or more will play to their potential and steal the tournament from the lower-average, more consistent players. The handicap system makes the tournament results fairer, in the sense that erratic and consistent are more equally likely to win.

Laurels to Hardy

Hardy's motivation in developing his golf model is not at all clear. This adventure in applied mathematics was uncharacteristic of his work. However, an interest in sports was not unusual. He was a devoted cricket fan, keeping up with the statistics and attending matches. So, we can imagine a question arising in conversation about the relative merits of consistency and erratic brilliance in golf. While Hardy's model is an oversimplification of golfing results, it provides insights into the role of variance in different types of golfing competitions. The insights include a clarification of the goals of the USGA handicap system, which does a better job of evening the odds in tournament play than in head-to-head competition.

References

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Biography.
ROLAND MINTON received his Ph.D. in 1982 from Clemson University, home of the 2003 NCAA golf national champions. He has taught at Roanoke College since 1986. He has co-authored with Bob Smith a series of calculus textbooks, and explored a number of fun mathematics applications ranging from sports of all kinds to chaos theory to Elvis, the calculus dog. His wife Jan also teaches at Roanoke College, his daughter Kelly teaches science in Kyle, Texas, and his son Greg is a graduate student at MIT. Roanoke College, Salem, Virginia. Email: minton@roanoke.edu.