

## PROOF OF THE SIMON-ANDO THEOREM

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ABSTRACT. In 1961, Simon and Ando wrote a classical paper describing the convergence properties of nearly completely decomposable matrices. Basically, their work concerned a partitioned stochastic matrix e.g.

$$A = \begin{bmatrix} A_1 & E_1 \\ E_2 & A_2 \end{bmatrix}$$

where  $A_1$  and  $A_2$  are square blocks whose entries are all larger than those of  $E_1$  and  $E_2$  respectively.

Setting

$$A^k = \begin{bmatrix} A_1^{(k)} & E_1^{(k)} \\ E_2^{(k)} & A_2^{(k)} \end{bmatrix},$$

partitioned as in  $A$ , they observed that for some, rather short, initial sequence of iterates the main diagonal blocks tended to matrices all of whose rows are identical, e.g.  $A_1^{(k)}$  to  $F_1$  and  $A_2^{(k)}$  to  $F_2$ . After this initial sequence, subsequent iterations showed that all blocks lying in the same column as those matrices tended to a scalar multiple of them, e.g.

$$\lim_{k \rightarrow \infty} A^k = \begin{bmatrix} \alpha F_1 & \beta F_2 \\ \alpha F_1 & \beta F_2 \end{bmatrix}$$

where  $\alpha \geq 0$  and  $\beta \geq 0$ .

The purpose of this paper is to give a qualitative proof of the Simon-Ando theorem.

Let  $B$  be an  $n \times n$  nonnegative matrix. Partition

$$B = \begin{bmatrix} B_1 & B_{12} & \cdots & B_{1r} \\ B_{21} & B_2 & \cdots & B_{2r} \\ \dots & \dots & \dots & \dots \\ B_{s1} & B_{s2} & \cdots & B_s \end{bmatrix}.$$

Choose any column of blocks, say

$$\bar{B}_k = \begin{bmatrix} B_{1k} \\ \vdots \\ B_k \\ \vdots \\ B_{sk} \end{bmatrix}.$$

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If, for all  $j$ , any entry in the  $j$ th column of  $\overline{B}_k$  which lies in  $B_k$  is at least as large as any entry in this column which lies in  $B_{ki}$ , for all  $i$ , then we say that  $B_k$  is block column dominant.

Throughout the paper we let

$$A = \begin{bmatrix} A_1 & \mathcal{E}_{12} & \cdots & \mathcal{E}_{1r} \\ \mathcal{E}_{21} & A_2 & \cdots & \mathcal{E}_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{E}_{r1} & \mathcal{E}_{r2} & \cdots & A_r \end{bmatrix}$$

be an  $n \times n$  stochastic matrix with each  $A_i$  being  $n_i \times n_i$  and column block dominant.

Let  $B$  be an  $n \times m$  nonnegative matrix. Define

$$\alpha_j(B) = \max_k b_{kj} - \min_k b_{kj}.$$

Thus,  $\alpha_j$  measures the spread of the entries in the  $j$ th column of  $B$ , and considering all  $j$ , the distance between the rows.

The next lemma measures the contraction of the spread under multiplication.

**Lemma 1.** *Let  $B$  be an  $n \times n$  nonnegative matrix and  $c$  an  $n \times 1$  nonnegative vector. Let  $\overline{r}$  and  $\underline{r}$  denote the largest and smallest row sums of  $B$ , respectively, and  $m$  the smallest entry in  $B$ . Then*

$$\alpha_1(Bc) \leq (\overline{r} - 2m)\alpha_1(c) + (\overline{r} - \underline{r}) \min_i c_i.$$

If  $\overline{r} \leq 1$ , then  $\max_i (Bc)_i \leq \max_i c_i$ .

*Proof.* First note that for any  $p$ ,

$$\begin{aligned} m \max_i c_i + \left( \sum_{k=1}^n b_{pk} - m \right) \min_i c_i \\ \leq \sum_{k=1}^n b_{pk} c_k \leq \left( \sum_{k=1}^n b_{pk} - m \right) \max_i c_i + m \min_i c_i. \end{aligned}$$

Of course if  $\overline{r} \leq 1$  the right side is bounded by  $\max_i c_i$ . For convenience, set  $Bc = c'$ . Then using the inequality above to measure spread,

$$\begin{aligned} \max_i c'_i - \min_i c'_i &\leq [(\overline{r} - m) \max_i c_i + m \min_i c_i] - [m \max_i c_i + (\underline{r} - m) \min_i c_i] \\ &\leq (\overline{r} - 2m) \max_i c_i - (\underline{r} - 2m) \min_i c_i \\ &\leq (\overline{r} - 2m)\alpha_1(c) + (\overline{r} - \underline{r}) \min_i c_i. \quad \square \end{aligned}$$

As in [3], for any nonnegative matrix  $C$ , define

$$\mathcal{T}(C) = \frac{1}{2} \max_{i,j} \sum_k |c_{ik} - c_{jk}|$$

called a coefficient of ergodicity of  $C$ . If

$$\mathcal{E}_p = (\mathcal{E}_{p1}, \dots, \mathcal{E}_{pp-1}, \mathcal{E}_{pp+1}, \dots, \mathcal{E}_{pr}),$$

the  $\mathcal{E}_{ij}$  blocks taken from  $A$ , then define

$$\varepsilon = \max_p \mathcal{T}(\mathcal{E}_p).$$

In addition, for the remainder of the paper, we let  $\bar{r}$  and  $\underline{r}$  denote the largest and smallest row sums of  $A_i$  and  $m$  the smallest entry in  $A_i$ , over all  $i$ . And also for the remainder of the paper we denote by  $B$  an  $n \times n$  stochastic matrix

$$B = \begin{bmatrix} B_1 & \delta_{12} & \cdots & \delta_{1r} \\ \delta_{21} & B_2 & \cdots & \delta_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ \delta_{r1} & \delta_{r2} & \cdots & B_r \end{bmatrix}$$

partitioned compatibly with  $A$ . The role of  $B$  in our iteration work is that if we write  $A^{k+1} = AA^k$ , then  $B$  will denote  $A^k$ .

The next two theorems are concerned with the contraction of the rows in all blocks of  $AB$ . Without loss of generality, we consider the 1, 1-block of this product. A preliminary lemma is needed.

**Lemma 2.**  $\alpha_j(A_1B_1 + \sum_{k>1} \mathcal{E}_{1k}\delta_{k1}) \leq (\bar{r} - 2m)\alpha_j(B_1) + (\bar{r} - \underline{r}) + \varepsilon$ .

*Proof.* Note that

$$\begin{aligned} & \alpha_j \left( A_1B_1 + \sum_{k>1} \mathcal{E}_{1k}\delta_{k1} \right) \\ & \leq \alpha_j(A_1B_1) + \alpha_j \left( \sum_{k>1} \mathcal{E}_{1k}\delta_{k1} \right) \\ & \leq \alpha_j(A_1B_1) + \max_p \sum_{r,k>1} \varepsilon_{pr}^{(1,k)} \delta_{rj}^{(k,1)} - \min_q \sum_{r,k>1} \varepsilon_{qr}^{(1,k)} \delta_{rj}^{(k,1)} \end{aligned}$$

where  $\varepsilon_{pr}^{(1,k)}$  and  $\delta_{rj}^{(k,1)}$  are entries in  $\mathcal{E}_{1k}$  and  $\delta_{k1}$ , respectively, with  $\varepsilon_{pr}^{(1,k)}$  and  $\delta_{rj}^{(k,1)}$  lying in rows  $p$  and  $r$  and columns  $r$  and  $j$ , respectively. Thus, using Lemma 1,

$$\begin{aligned} \alpha_j \left( A_1B_1 + \sum_{k>1} \mathcal{E}_{1k}\delta_{k1} \right) & \leq \alpha_j(A_1B_1) + \max_{p,q} \sum_{r,k>1} |\varepsilon_{pr}^{(1,k)} - \varepsilon_{qr}^{(1,k)}| \delta_{rj}^{(k,1)} \\ & \leq \alpha_j(A_1B_1) + \varepsilon \\ & \leq (\bar{r} - 2m)\alpha_j(B_1) + (\bar{r} - \underline{r}) + \varepsilon. \end{aligned}$$

The result follows.  $\square$

**Theorem 1.** Let  $\gamma$  be a nonnegative number with  $\gamma < 1$ . If  $\frac{\bar{r} - \underline{r} + \varepsilon}{\gamma - \bar{r} + 2m} \leq \alpha_j(B_1)$ , then

$$\alpha_j \left( A_1B_1 + \sum_k \mathcal{E}_{1k}\delta_{k1} \right) \leq \gamma \alpha_j(B_1).$$

*Proof.* By hypothesis,

$$\frac{\bar{r} - \underline{r} + \varepsilon}{\gamma - \bar{r} + 2m} \leq \alpha_j(B_1).$$

Thus,

$$\bar{r} - \underline{r} + \varepsilon \leq (\gamma - \bar{r} + 2m)\alpha_j(B_1).$$

Hence,

$$(\bar{r} - 2m)\alpha_j(B_1) + (\bar{r} - \underline{r}) + \varepsilon \leq \gamma \alpha_j(B_1).$$

And by Lemma 2,

$$\alpha_j \left( A_1 B_1 + \sum_k \mathcal{E}_{1k} \delta_{k1} \right) \leq \gamma \alpha_j(B_1).$$

Note, if  $\gamma$  is chosen smallest in Theorem 1, then

$$\gamma = \frac{(\bar{r} - \underline{r} + \varepsilon) + (\bar{r} - 2m)\alpha_j(B_1)}{\alpha_j(B_1)},$$

which is decreasing in  $\alpha_j(B_1)$ . So as  $\alpha_j(B_1)$  gets smaller,  $\gamma$  gets bigger.

On the other hand we have the following.  $\square$

**Theorem 2.** *Let  $\gamma$  be a nonnegative number with  $\gamma < 1$ . If  $\frac{\bar{r} - \underline{r} + \varepsilon}{\gamma - \bar{r} + 2m} > \alpha_j(B_1)$ , then*

$$\alpha_j \left( A_1 B_1 + \sum_k \mathcal{E}_{1k} \delta_{k1} \right) \leq \gamma \left( \frac{\bar{r} - \underline{r} + \varepsilon}{\gamma - \bar{r} + 2m} \right).$$

*Proof.* By hypothesis,

$$\frac{\bar{r} - \underline{r} + \varepsilon}{\gamma - \bar{r} + 2m} > \alpha_j(B_1).$$

Then by Lemma 2,

$$\begin{aligned} \alpha_j \left( A_1 B_1 + \sum_k \mathcal{E}_{1k} \delta_{k1} \right) &\leq (\bar{r} - 2m)\alpha_j(B_1) + (\bar{r} - \underline{r}) + \varepsilon \\ &\leq (\bar{r} - 2m) \left( \frac{\bar{r} - \underline{r} + \varepsilon}{\gamma - \bar{r} + 2m} \right) + (\bar{r} - \underline{r}) + \varepsilon \\ &\leq (\bar{r} - \underline{r} + \varepsilon) \left[ \frac{\bar{r} - 2m}{\gamma - \bar{r} + 2m} + 1 \right] \\ &\leq (\bar{r} - \underline{r} + \varepsilon) \left( \frac{\bar{r} - 2m + \gamma - \bar{r} + 2m}{\gamma - \bar{r} + 2m} \right) \\ &\leq (\bar{r} - \underline{r} + \varepsilon) \left( \frac{\gamma}{\gamma - \bar{r} + 2m} \right). \end{aligned}$$

Let  $a = \frac{\bar{r} - \underline{r} + \varepsilon}{\gamma - \bar{r} + 2m}$ . Putting Theorem 1 and Theorem 2 together we see that if  $\alpha_j(B_1) \notin [0, a]$ , then the next iterate  $AB$ , after  $B$ , shows contraction of the rows  $A_1 B_1 + \sum_k \mathcal{E}_{1k} \delta_{k1}$ . On the other hand if  $\alpha_j(B_1) \in [0, a]$ , then so is  $\alpha_j(A_1 B_1 + \sum_k \mathcal{E}_{1k} \delta_{k1})$ , the next iterate. Thus, under iteration, all blocks tend to flat matrices and when close remain so.

We now consider a fixed rate of convergence. Set

$$\gamma = \bar{r} - m.$$

Theorem 1 says that if

$$\frac{\bar{r} - \underline{r} + \varepsilon}{m} \leq \alpha_j(B_1),$$

then

(i)  $\alpha_j(A_1 B_1 + \sum_k \mathcal{E}_{1k} \delta_{k1}) \leq (\bar{r} - m)\alpha_j(B_1)$ , so there is further contraction of rows in the 1,1-block of  $AB$ , and this block continues the trend toward a flat matrix.

By Theorem 2, if

$$\frac{\bar{r} - \underline{r} + \varepsilon}{m} > \alpha_j(B_1),$$

then

$$(ii) \quad \alpha_j \left( A_1 B_1 + \sum_k \mathcal{E}_{1k} \delta_{k1} \right) \leq (\bar{r} - m) \left( \frac{\bar{r} - \underline{r} + \varepsilon}{m} \right) \\ \leq \frac{\bar{r} - \underline{r} + \varepsilon}{m},$$

so at this time, this block remains close to a flat matrix, other blocks are handled in the same way.

We have now shown that after some initial sequence of products, say  $A, A^2, \dots, A^s$ , the blocks of  $A^s$  are nearly flat, and in the remaining products  $A^{s+1}, A^{s+2}, \dots$ , these blocks stay nearly flat. We now show, in addition, that the blocks, which lie in the same block column, stay almost proportional to the corresponding main diagonal block in  $A^s$ .

For this work we need a preliminary lemma. We use that  $e$  is a column vector all of whose entries are 1.  $\square$

**Lemma 3.** *Suppose, for all  $i$  and  $j$ , the largest entry in  $\mathcal{E}_{ij}e$  is less than  $\alpha$ . Set  $B = A^k$ . Then the entries in  $\delta_{ij}e$  are less than  $k\alpha$ .*

*Proof.* The proof is by induction on  $k$ . For  $k = 1$ , the result is obvious.

Suppose the result holds for  $k = \bar{k}$ . Set  $B = A^{\bar{k}}$ . Then  $A^{\bar{k}+1} = AB$ . A typical off-diagonal block is

$$A_i \delta_{ij} + \sum_{r \neq i, j} \mathcal{E}_{ir} \delta_{rj} + \mathcal{E}_{ij} B_j.$$

Note that

$$\left( A_i \delta_{ij} + \sum_{r \neq i, j} \mathcal{E}_{ir} \delta_{rj} + \mathcal{E}_{ij} B_j \right) e \\ = A_i \delta_{ij} e + \sum_{r \neq i, j} \mathcal{E}_{ir} \delta_{rj} e + \mathcal{E}_{ij} B_j e \\ \leq \bar{k} \alpha A_i e + \bar{k} \alpha \sum_{r \neq i, j} \mathcal{E}_{ir} e + \mathcal{E}_{ij} e \\ \leq \bar{k} \alpha \left( A_i + \sum_{r \neq i, j} \mathcal{E}_{ir} \right) e + \mathcal{E}_{ij} e;$$

since  $A$  is stochastic, this is no greater than

$$(\bar{k} \alpha) e + \alpha e \leq (\bar{k} + 1) \alpha e. \quad \square$$

This lemma gives a bound on the size of the off-diagonal blocks of  $A^s$ . When these blocks are sufficiently small, we can prove the theorem about the blocks of future iterates being nearly proportional to the main diagonal block in  $A^s$ , lying in its column.

**Theorem 3.** Let  $\bar{\varepsilon} > 0$  be given. Suppose that after  $s$  iterates the rows in the main diagonal blocks of  $A^s$  are within  $\bar{\varepsilon}$  and the entries in the off-diagonal blocks are no larger than  $\bar{\varepsilon}$ . Let  $f_i$  be the average of the rows of the  $i$ th main diagonal block of  $A^s$  and  $F_{ij}$  the  $n_i \times n_j$  matrix whose rows are  $f_i$ . Then, for all  $k \geq s$ , there are diagonal matrices  $D_{ij}^{(k)}$  and a matrix  $E_k$  such that

$$A^k = \begin{bmatrix} D_{11}^{(k)} F_{11} & D_{12}^{(k)} F_{12} & \cdots & D_{1r}^{(k)} F_{1r} \\ D_{21}^{(k)} F_{21} & D_{22}^{(k)} F_{22} & \cdots & D_{2r}^{(k)} F_{2r} \\ \dots & \dots & \dots & \dots \\ D_{r1}^{(k)} F_{r1} & D_{r2}^{(k)} F_{r2} & \cdots & D_{rr}^{(k)} F_{rr} \end{bmatrix} + E_k$$

where the entries in  $E_k$  are no larger than  $\bar{\varepsilon}$ .

*Proof.* By the hypothesis,

$$A^s = \begin{bmatrix} F_{11} & \bar{\varepsilon} F_{12} & \cdots & \bar{\varepsilon} F_{1r} \\ \bar{\varepsilon} F_{21} & F_{22} & \cdots & \bar{\varepsilon} F_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\varepsilon} F_{r1} & \bar{\varepsilon} F_{r2} & \cdots & \bar{\varepsilon} F_{rr} \end{bmatrix} + E_s$$

where  $E_s$  has entries no larger than  $\bar{\varepsilon}$ . Now,

$$A^{s+1} = \begin{bmatrix} (\mathcal{E}_{i1} \cdots \mathcal{E}_{ii-1} A_i \mathcal{E}_{ii+1} \cdots \mathcal{E}_{ir}) \begin{pmatrix} \bar{\varepsilon} F_{1j} \\ \vdots \\ \bar{\varepsilon} F_{ij} \\ F_{jj} \\ \bar{\varepsilon} F_{i+1j} \\ \vdots \\ \bar{\varepsilon} F_{rj} \end{pmatrix} \end{bmatrix} + A E_s$$

where the  $ij$ th block is displayed in the first matrix of the right side. This block can be written as

$$(\bar{\varepsilon} \mathcal{E}_{i1} \cdots \bar{\varepsilon} \mathcal{E}_{ii-1} A_i \bar{\varepsilon} \mathcal{E}_{ii+1} \cdots \bar{\varepsilon} \mathcal{E}_{ir}) \begin{pmatrix} F_{1j} \\ \vdots \\ F_{ij} \\ F_{jj} \\ F_{i+1j} \\ \vdots \\ F_{rj} \end{pmatrix}.$$

Since the columns of the  $F_{ij}$ 's are identical, we can write this matrix as  $D_{ij}^{(s+1)} F_j$  where  $D_{ij}^{(s+1)}$  is a diagonal matrix whose  $t$ th main diagonal entry is the  $t$ th row sum of  $(\bar{\varepsilon} \mathcal{E}_{i1} \cdots \bar{\varepsilon} \mathcal{E}_{ii-1} A_i \bar{\varepsilon} \mathcal{E}_{ii+1} \cdots \bar{\varepsilon} \mathcal{E}_{ir})$ .

Since  $A$  is stochastic,  $E_{s+1} = A E_s$  has entries no larger than  $\bar{\varepsilon}$ . Thus,  $A^{s+1}$  has the form of the theorem. Now, duplicating the argument above shows that  $A^k$  has the form described in the theorem for all  $k \geq s$ .  $\square$

Putting the results together we have the following. Theorem 1 and Theorem 2 assure that during some initial sequence  $A^1, A^2, \dots, A^s$  all blocks tend toward being nearly flat and then stay nearly flat. For a convergence rate of  $\bar{r} - m$ ,  $\frac{\bar{r} + r + \varepsilon}{m}$



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