\textbf{L-CLASSES ON PSEUDOMANIFOLDS WITH ONE SINGULAR STRATUM}

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Abstract. We study the index theorem and Chern character of an admissible pseudomanifold $X^\dagger$ with one singular stratum. Under a condition on the link, we give a de Rham type realization of the Goresky-MacPherson-Siegel $L$-classes on $X^\dagger$ in terms of curvature forms and eta invariant of the link.

0. Introduction

In [CST] and [MW1] Connes-Sullivan-Teleman and Moscovici-Wu solved a long-standing problem of recovering the topological Pontryagin classes from local data. In [MW1] and [MW2], using the finite propagation speed property in the unbounded picture, Moscovici-Wu gave a realization of the $L$-classes, hence of the Pontryagin classes, of the topological manifold in terms of Alexander-Spanier cycles by looking at the "straight" Chern character of [CM].

In this paper, we will generalize the result in [MW1] to a particular type of pseudomanifolds, namely $X^\dagger = M \cup (c^L(L) \times N)$. As in [C1] and [GM], for spaces with singularities, we work with "characteristic classes" in homology rather than cohomology. In [GM] Goresky and MacPherson defined $L$-classes for pseudomanifolds with even codimensional strata. By the work of Siegel [Si], one can extend this definition to Witt spaces. Our approach differs from [MW1] in the sense that we relate the "straight" Chern character of $D$ to $L$-class directly. This asserts that (Theorem 3.4) the Goresky-MacPherson-Siegel $L$-class of $X^\dagger$ (with $2L$ having zero oriented cobordism) is represented by the following cycle:

$$L_*(X^\dagger; \rho_{X^\dagger}) = 2^{2m'} L_{m'}(R(g^M)) \oplus -\delta_L \eta(L) 2^{2n'} L_{n'}(R(g^N)),$$

where $m' = \frac{m-4}{2}$, $n' = \frac{n-4}{2}$ and $\delta_L = \frac{1}{2}(1 - (-1)^{\ell})$. Furthermore, we will show that the Chern character of $D$ coincides with the "straight" Chern character of $D$.

In [BC] and [C2], Bismut and Cheeger defined $L$-classes by means of the index pairing. Theorem 4.1 below shows that the Bismut-Cheeger $L$-classes and Goresky-MacPherson-Siegel $L$-classes are the same (up to constants) for spaces $X^\dagger$ defined above.
1. Preliminaries

1.1. Pseudomanifolds. For your reference, let’s recall some definitions in [C1]. An \( m \)-dimensional pseudomanifold \( Y^m \) is a finite simplicial complex such that every point is contained in a closed \( m \)-simplex and every \((m-1)\)-simplex is a face of either one or two \( m \)-simplices. We also endow \( Y^m \) with a metric, which determines a distance function \( \rho_Y \), and assume that \( Y \setminus \sum^{m-2} \) is a flat manifold (in the induced metric) where \( \sum^{m-2} \) is the \((m-2)\)-skeleton associated with a triangulation. A pseudomanifold is called admissible [C1, p.127] if the middle \( L^2 \)-cohomology group of even dimensional links, in the Riemannian handle decomposition, vanishes.

After the construction of Fredholm module in Section 2, we will consider a particular type of pseudomanifolds:

Let \( M \) be a smooth, oriented, compact and connected \( m \)-dimensional manifold with boundary \( \partial M = L \times N \) where \( L \) and \( N \) are smooth, oriented and compact manifolds (without boundary) of dimensions \( \ell \) and \( n \) respectively, and either \( \ell \) is odd or \( H^2_*(L) = 0 \).

In this paper, we will use the following notation:

\[
\begin{align*}
c_{0,\infty}(L) &= (0, \infty) \times L, & \text{infinite cone with link } L, \\
c(L) &= (0,1) \times L, & \text{cone with link } L, \\
\dagger &= \text{the tip of the cone,} \\
c^\dagger(L) &= c(L) \cup \{ \dagger \}, & \text{completed cone with link } L, \\
X &= M \cup (c(L) \times N), & \text{regular part,} \\
X^\dagger &= M \cup (c^\dagger(L) \times N), & \text{pseudomanifold with one singular stratum.}
\end{align*}
\]

We will endow \( X^\dagger \) with a metric \( g \) (not necessary flat on \( X \)) such that:

(i) \( g \) is a measurable metric on \( X^\dagger \);
(ii) \( g|_M \) is a smooth metric on \( M \) and is a product near \( \partial M \);
(iii) \( g|_{c(L) \times N} = (dr^2 + \psi(r)^2 g^L) \oplus g^N \), where \( g^L \) and \( g^N \) are smooth metrics on \( L \) and \( N \) respectively, and \( \psi : [0,1] \to [0,1] \) is a \( C^\infty \) function such that

\[
\psi(r) = \begin{cases} r, & r \in [0, \frac{2}{3}], \\
1, & r \in [\frac{2}{3}, 1], \end{cases}
\]

and \( \psi(r) \neq 0 \) for \( r > 0 \).

Clearly, this is a Lipschitz metric.

In order not to change the metric (i.e. \( \rho_{X^\dagger}|_{X \times X} = \rho_X \)), \( L \) must be connected. Otherwise, we will need to add a cone on each component of \( L \). In other words, we have to consider the normalization of the space [GM, p.151]. So in the rest of this paper, we will assume that \( L \) and \( N \) are connected. In this case, \( X^\dagger \) is the metric completion of \( X \).

1.2. Sullivan complex. In the presence of singularities it is more convenient to use the Sullivan complex ([BC], [MW2] and [Su]). Let’s recall the corresponding results [MW2] for the space \( X^\dagger \). Let \( pr : c(L) \times N \to N \) be the projection onto the second factor and \( j : L \times N \to M \) be the inclusion map. The complex of stratified differential forms on \( X^\dagger \) is:

\[
\Omega^*(X^\dagger)_{SA} = \{ \omega \in \Omega^*(X) : \omega|_{c(L) \times N} = pr^*(\tilde{\omega}), \tilde{\omega} \in \Omega^*(N) \}.
\]
\( \Omega^0(X^1)_{SA} \) is an algebra, which will also be denoted by \( C^\infty_{SA}(X^1) \). The differentials are the usual ones. From [MW2], \( H^*(X^1) \cong H_*(\Omega^*(X^1)_{SA}, d) \). For homology, there is also a chain complex of de Rham type, namely
\[
\Omega_q(X^1)_{SA} = \Omega^{m-q}(M) \oplus \Omega^{n-q}(N)
\]
with the boundary operator \( \partial : \Omega_q(X^1)_{SA} \to \Omega_{q-1}(X^1)_{SA} \) given by
\[
\partial(\omega_1, \omega_2) = \left( (-1)^q d\omega_1, (-1)^q d\omega_2 + \int_L j^* \omega_1 \right).
\]
Again from [MW2], \( H_*(X^1) \cong H_*(\Omega_*(X^1)_{SA}, \partial) \).

For these complexes, there is a pairing
\[
\Omega_q(X^1)_{SA} \otimes \Omega^q(X^1)_{SA} \longrightarrow \mathbb{C},
\]
\[
(a, b) \otimes c \longrightarrow \int_M a \wedge c + \int_N b \wedge c,
\]
which induces a pairing on the corresponding homology and cohomology.

### 1.3. Goresky-MacPherson-Siegel \( \mathcal{L} \)-class

Recall that [Si, p.1068] a pseudo-manifold is a Witt-space if every even dimensional link \( L^\ell \) of an odd codimensional intrinsic stratum satisfies \( IH^m_{\frac{\ell}{2}}(L; \mathbb{Q}) = 0 \). Let \( V \) be a Witt-space which is also a compact subset of some \( C^\infty \)-manifold \( V' \). If \( \dim V \neq 4k \), then we define the signature \( \sigma \) of \( V \) to be 0. If \( \dim V = 4k \), then let \( \tilde{m} = (0, 0, 1, 1, 2, 2, \ldots, 2k - 1) \). By [Si, Theorem 3.4], there is a non-degenerate rational pairing
\[
IH^m_{i}(V; \mathbb{Q}) \times IH^m_{j}(V; \mathbb{Q}) \to \mathbb{Q}
\]
for \( i + j = \dim V, \ i, j \geq 0 \). In this case, the signature \( \sigma \) of \( V \) is defined as the signature of the associated quadratic form.

As in [GM, p.158], a continuous map \( f : V \to S^k \) is called transverse if

- \((a)\): \( f \) is the restriction of a \( C^\infty \)-map \( \tilde{f} : U \to S^k \) for some neighborhood \( U \) of \( V \) in \( V' \),
- \((b)\): \( \tilde{f} \) is transverse regular to the north pole \( p \in S^k \),
- \((c)\): \( \tilde{f}^{-1}(p) \) is transverse to each stratum of \( V \).

As there is such a representative in each homotopy class and the signature is cobordism invariant [GM, p.158], [Si, Theorem 2.1], one can define
\[
\theta : [V, S^k] \longrightarrow \mathbb{Z},
\]
\[
[f] \longrightarrow \sigma(f^{-1}(p)).
\]
The Goresky-MacPherson-Siegel \( \mathcal{L} \)-class, \( \mathcal{L}_k(V) \in H_k(V; \mathbb{R}) \), is defined as the homomorphism
\[
\theta \otimes I : H^k(V; \mathbb{R}) \to \mathbb{R}
\]
where we have identified \( [V, S^k] \otimes \mathbb{R} \cong H^k(V; \mathbb{R}) \) when \( 2k > m + 1 \).

We can remove the assumption \( 2k > m + 1 \) by crossing \( V \) with a sphere as in [GM, p.158] and [MS].
2. SIGNATURE OPERATOR AND K-CYCLE

2.1. Finite propagation speed. Let $Y$ be an admissible Riemannian pseudomanifold. In this subsection, the domains of the operators are:

$$\text{Dom}(d) = \{ \alpha \in \Gamma^\infty : \alpha, d\alpha \in L^2 \},$$
$$\text{Dom}(\delta) = \{ \alpha \in \Gamma^\infty : \alpha, \delta \alpha \in L^2 \}.$$

In the next subsection, we will show that it does not matter which domain we use.

By [C1], $d^* = \delta$. Then by [H1, Lemma 4.3], $D := d + \delta$ is self-adjoint with domain $\text{Dom}(d) \cap \text{Dom}(\delta)$.

Let $f \in C(Y)$ act on $L^2(\wedge^* T(Y \setminus \Sigma))$ by multiplication.

Lemma 2.1. Let $h \in C^{Lip}(Y)$; then $h \cdot \text{Dom}(D) \subset \text{Dom}(D)$.

Proof. Let $\omega \in \text{Dom}(D)$ and $J_\epsilon$ be the mollifier corresponding to a bump function $\Phi$. So $J_\epsilon(h\omega)$ is smooth and $J_\epsilon(h\omega) \overset{L^2}{\rightarrow} h\omega$. Since

$$d(h\omega) = dh \wedge \omega + h d\omega,$$
$$\delta(h\omega) = *(dh \wedge *\omega) + h \delta \omega,$$

$$\|f\|_\infty, \|dh\|_\infty \leq 1, \omega, d\omega, \delta \omega \in L^2$$
and by using partition of unity, we have

$$J_\epsilon(h\omega)(x) = \frac{1}{\epsilon^n} \int \Phi\left(\frac{x - y}{\epsilon}\right)h(y)\omega(y)dy$$

$$= \frac{1}{\epsilon^n} \int \Phi\left(\frac{y}{\epsilon}\right)h(x - y)\omega(x - y)dy$$
in local coordinate patch.

Then $J_\epsilon(h\omega), d(J_\epsilon(h\omega)), \delta(J_\epsilon(h\omega)), d(h\omega), \delta(h\omega) \in L^2$.

By Friedrichs Lemma [T, p.114],

$$dJ_\epsilon(h\omega) \overset{L^2}{\rightarrow} dh, \quad \delta J_\epsilon(h\omega) \overset{L^2}{\rightarrow} \delta(h\omega).$$

Hence $h\omega \in \text{Dom}(D)$.

Proposition 2.2. $D$ has finite propagation speed with respect to $C(Y)$, i.e.

$$\forall t \in \mathbb{R}, \quad \text{supp}(e^{itD}) \subset \{ (x, y) \in Y \times Y : \rho_Y(x, y) \leq |t| \}.$$

Also, let $f \in \mathcal{S}(\mathbb{R})$ such that $\text{supp} \hat{f} \subset [-\alpha, \alpha]$ for some $\alpha > 0$; then

$$\text{supp}(f(D)) \subset \{ (x, y) \in Y \times Y : \rho_Y(x, y) \leq \alpha \}.$$

Proof. As $\rho_Y$ is a Lipschitz function on $Y \setminus \sum^{m-2}$, the results follow from [H2, Lemma 1.10] as in [H2, Corollary 1.11].

In the remainder of this paper, we will assume $m$ to be even, unless otherwise stated.

Remark 2.3. So far, the results for $Y$ in this section are still true when $Y$ endows a metric which is quasi-isometric to a flat metric.
2.2. Essential self-adjointness of signature operator. Due to some technicalities in the rest of this paper, we will investigate the essential self-adjointness of the twisted signature operator on $X^\dagger$. Assume $\partial X$ is even; then $\partial X = L \times N$ is odd dimensional. Let $(E, \nabla)$ be a Hermitian vector bundle on $X$ with a unitary connection such that its restriction to $c(L) \times N$ is pulled back from $N$.

We will consider the scaling of the metric in the conical direction as follows:
\[ g|_{c(L) \times N} = \left( \frac{dr^2}{\epsilon} + \psi(r^2)g^L \right) \oplus g^N \]
for $\epsilon > 0$.

In the following we will use the standard domain, specifically $\text{Dom}(D) = \{ \alpha \in \Gamma_c^\infty(X) \}$.

To study the self-adjointness of a operator $D$, we will examine the deficiency indices $n_\pm(D) = \dim \text{Ker}(D \mp iI)$. Let us recall a proposition from [L1].

**Proposition 2.4** ([L1], Corollary 2.2). Let $M_i$ be an oriented Riemannian manifold and $D_i$ be a generalized Dirac operator over $M_i$, $i = 1, 2$. Let $U_i$ be an open subset of $M_i$, $i = 1, 2$. Suppose $M_1 \setminus U_1$ and $M_2 \setminus U_2$ are complete manifolds with compact boundary and there exists an isometry from $\gamma : U_1 \to U_2$ which lifts to an isomorphism of Clifford structure. Then $n_\pm(D_1) = n_\pm(D_2)$.

For easy reference, we will add subscripts to operators to indicate the underlying manifold and vector bundle.

**Proposition 2.5.** There exists $\delta > 0$ (independent of $E$) such that $\forall \epsilon \in (0, \delta)$, the twisted signature operator $D_{X,E}$ is essentially self-adjoint.

**Proof.** We will divide this into two cases.

Case 1: $\ell$ is odd and $n$ is even.

On $c(L) \times N$, $D_{X,E} \simeq D_{c(L)} \hat{\otimes} I + I \hat{\otimes} D_{N,E}$.

For sufficiently small $\epsilon$, by [BS, Lemma 5.4], $D_{c_0,\infty(L)}$ is essentially self-adjoint. Hence, by [RS, Theorem VIII.33], $D_{c_0,\infty(L) \times N,E}$ is essentially self-adjoint. Therefore,
\[ n_\pm(D_{c_0,\infty(L) \times N,E}) = 0. \]

Then by Proposition 2.4,
\[ n_\pm(D_{X,E}) = n_\pm(D_{c_0,\infty(L) \times N,E}) = 0. \]

Therefore, $D_{X,E}$ is essentially self-adjoint for sufficiently small $\epsilon$.

Case 2: $\ell$ is even and $n$ is odd.

Since $\dim(c_{0,\infty}(L))$ is odd, the signature operator splits as
\[ D_{c_{0,\infty}(L)} = D^+_{c_{0,\infty}(L)} \oplus D^-_{c_{0,\infty}(L)}. \]

Let $\lambda_{\omega_L}$ be the Clifford multiplication by $\sqrt{t}\lambda e_1 \cdots e_l$ where $(e_1, \ldots, e_l)$ is an oriented orthonormal frame for $TL$.

By [L1, Lemma 1.2, Proposition 4.1, Proposition 5.3],
\[ n_\pm(D^+_{c_0,\infty(L)}) = n_\mp(D^-_{c_0,\infty(L)}) = \dim(\ker(\lambda_{\omega_L} \mp I) \cap \ker P) + \sum_{0 < \lambda < \frac{1}{2}} \dim \ker(P - \lambda I) \]
for some operator $P$. 

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By [BL, Lemma 2.2], Ker$P = H^\frac{1}{2}(L) = 0$.
By [BL, Corollary 2.3],
\[ \sum_{0 < \lambda < \frac{1}{4}} \dim \text{Ker}(P - \lambda I) = 0 \]
for sufficiently small $\epsilon$.

Thus, $D_{c_{0,\infty}(L)}^\pm$ is essentially self-adjoint and so is $D_{c_{0,\infty}(L)}$. Notice that
\[ D_{c_{0,\infty}(L) \times N,E} = D_{c_{0,\infty}(L)} \otimes I + \phi \otimes D_{N,E}, \]
where $\phi = \begin{cases} 1 & \text{on } \Omega_{\text{even}}, \\ -1 & \text{on } \Omega_{\text{odd}}. \end{cases}$

Then as in the proof of [RS, Theorem VIII.33], $D_{c_{0,\infty}(L) \times N,E}$ is essentially self-adjoint. Thus,
\[ n_{\pm}(D_{c_{0,\infty}(L) \times N,E}) = 0. \]
By Proposition 2.4,
\[ n_{\pm}(DX,E) = n_{\pm}(D_{c_{0,\infty}(L) \times N,E}) = 0. \]

Hence the result follows.

2.3. Singular elliptic estimate. In the remaining of this paper, we will assume a scaling on the conical metric such that there are unique self-adjoint extensions for the signature operator on $X$ and the twisted signature operator on $X$ as well as the twisted signature operator on $c_{0,\infty}(L) \times N$. These facts will be used to remove the effect of small eigenvalues. By abuse of notation, we will use the same symbol to denote the self-adjoint extensions of the operators. As in Section 2.2, $\dim X$ is even.

Let $(E, \nabla)$ be a Hermitian vector bundle on $X$ with unitary connection such that its restriction to $c(L) \times N$ is pulled back from $N$.

To furnish the computation on heat kernels, we need to recall a definition from [L2]. Let $\mathcal{M}$ be a Riemannian manifold and $U \subset \mathcal{M}$ an open subset with compact boundary. Let $P_0$ be a symmetric differential operator of order $\mu$ defined on a bundle over $\mathcal{M}$. Assume there exists a closed self-adjoint extension $P$ of $P_0$ such that $\text{Dom}(P)$ is invariant under multiplication by functions $\varphi \in C_0^\infty(\mathcal{M})$ satisfying $\varphi|_{U_0} \equiv 1$. (This is always true if $\mathcal{M} \setminus U$ is compact.)

Let $KD(P, U) := \{ s \in \text{Dom}(P) : \supp s \subset U, \ \text{dist}(\supp s, \partial U) > 0 \}$.

**Definition 2.6** ([L2], p.41). $P$ satisfies the singular elliptic estimate (SE) on $U$ if $\exists \varrho \in L^2_{\text{loc}}(\mathcal{M}) \cap C(\mathcal{M}), \varrho > 0, \varrho|_U \in L^2(U)$ and $\ell \in \mathbb{R}^+$ such that for $x \in U$ and $s \in KD(P^\ell, U),
\[ |s(x)| \leq \varrho(x) \left( ||s||_{L^2(U,E)} + ||P^\ell s||_{L^2(U,E)} \right). \]

The importance of this concept lies in the following theorem.

**Theorem 2.7** ([L2]). Let $\mathcal{M}_i, U_i, P_{i,0}$ and $P_i, i = 1,2$ as above. Assume there is an isometry $F : U_1 \to U_2$, which lifts to a bundle isometry $F_* : E_1|_{U_1} \to E_2|_{U_2}$ such that $P_{1,0} = F_*^{-1} \circ P_{2,0} \circ F_*$. We will identify $U_1$ with $U_2$ and denote it by $U$. We choose an open subset $W \subset U$, with smooth compact boundary such that $\overline{W} \subset U$ and $U \setminus \overline{W}$ is relatively compact. If $P_1$ and $P_2$ satisfy (SE) over $W$, then $\forall N > 0, \exists C > 0$ such that for $x, y \in W$,
\[ \left| \left( P_1^k \right)^2 e^{-tP_2^2} (x, y) - \left( P_2^k \right)^2 e^{-tP_2^2} (x, y) \right| \leq C \varrho(x) \varrho(y) t^N \]
where $k \in \mathbb{Z}_+ \cup \{ 0 \}$.
In order to establish singular elliptic estimate for \( X \), we need to recall some notation from \([L2]\).

Let \( \tilde{\rho} : X \to \mathbb{R} \) be a smooth function such that
\[
\tilde{\rho}|_M = 1, \quad \tilde{\rho}((r,x,y)) = r \quad \text{on } c_{0,\frac{1}{2}}(L) \times N.
\]
Let \( \mathcal{H} \) be a Hilbert space, \( T > 0 \) be a self-adjoint operator on \( \mathcal{H} \) and
\[
\mathcal{D}^{\infty}(T) := \bigcap_{k \geq 1} \text{Dom}(T^k).
\]
For \( x, y \in \mathcal{D}^{\infty}(T) \) and \( s \in \mathbb{R} \),
\[
(x,y)_s := (T^s x, T^s y).
\]
Let \( \mathcal{H}_T^\prime \) be the completion of \( \mathcal{D}^{\infty}(T) \) with respect to \( \| \cdot \|_s \). Then by \([L2, \text{Lemma } 1.2.1]\), \( \Delta = \Delta_L = (r \frac{\partial}{\partial r})^2 + D^2_r \geq 1 \) and \( \Delta = \Delta_L \otimes I + I \otimes D^2_{N,E} \) is essentially self-adjoint on \( C^\infty_c(c_{0,\infty}(L)) \otimes C^\infty(N,E) \).

Define
\[
\mathcal{H}^{s,0}(c_{0,\infty}(L) \times N,E) := \mathcal{H}_L^s,
\]
\[
\mathcal{H}^{s,\gamma}(c_{0,\infty}(L) \times N,E) := r^{-\gamma} \mathcal{H}^{s,0}(c_{0,\infty}(L) \times N,E)
\]
with scalar product \( (f,g)_{s,\gamma} := (r^{-\gamma} f, r^{-\gamma} g)_s \).

Now let \( \Delta_{X,E} \) be a non-negative elliptic differential operator of order 2 on \( X \) such that
\[
\Delta_{X,E}|_M \geq c_1 \quad \text{for some } c_1 > 0
\]
and
\[
\Delta_{X,E}|_{c(L) \times N} = \Delta.
\]

By \([L2, \text{Corollary } 2.2]\), \( \Delta_{X,E} \) is essentially self-adjoint on \( C^\infty_c(X,E) \).

Define
\[
K^{s,0}(X,E) := \mathcal{H}_L^s,
\]
\[
K^{s,\gamma}(X,E) := \tilde{\rho}^{-\gamma} K^{s,0}(X,E)
\]
with scalar product \( (f,g)_{s,\gamma} := (\tilde{\rho}^{-\gamma} f, \tilde{\rho}^{-\gamma} g)_s \).

In \([L2]\) Lesch proved the following lemma in the case when \( N \) is equal to a point. We shall prove this in a more general context.

**Lemma 2.8.** There exist constants \( C \) and \( \mu > 0 \) such that
\[
\text{Dom}(D^m_{X,E}) \subset K^{m,\mu}(X,E)
\]
and \( \forall f \in \text{Dom}(D^m_{X,E}) \),
\[
\|f\|_{m,\mu} \leq C(\|f\|_{0,0} + \|D^m_{X,E} f\|_{0,0}).
\]

**Proof.** Let \( \varphi \in C^\infty_c(\mathbb{R}) \) such that \( \varphi \equiv 1 \) near 0.

By \([L2, \text{Proposition } 1.3.19]\), the case when \( N \) is equal to a point, there exists \( \mu > 0 \) such that \( \varphi \text{ Dom}(D^m_{c(L)}) \subset \mathcal{H}^{m,\mu}(c_{0,\infty}(L)) \). So,
\[
\varphi \text{ Dom}(D^m_{c(L)} \otimes \text{Dom}(D^m_{N,E}) \subset \mathcal{H}^{m,\mu}(c_{0,\infty}(L) \otimes N,E).
\]

Hence, \( \text{Dom}(D^m_{X,E}) \subset K^{m,\mu}(X,E) \).

The inequality in the lemma is now equivalent to the assertion that \( \text{Dom}(D^m_{X,E}) \hookrightarrow K^{m,\mu}(X,E) \) is continuous. This follows from Closed Graph Theorem and the fact that the Sobolev norms in \( \text{Dom}(D^m_{X,E}) \) and \( K^{m,\mu}(X,E) \) are stronger than the \( L^2 \)-norm. \( \square \)
Proposition 2.9. There exist constants $C$ and $\mu > 0$ such that $\forall f \in \text{Dom}(D^n_{X,E})$,

$$|f(x)| \leq C \hat{p}(x)^{\mu - \frac{1}{2}} (\|f\| + \|D^n_{X,E}f\|).$$

That is, $D_{X,E}$ satisfies the singular elliptic estimate (Definition 2.6).

Proof. Follows from Lemma 2.8 and the corresponding estimate in model cone [L2, Cor. 1.2.9].

2.4. The construction of $K$-cycle. Let $\mathcal{H} = L^2(\wedge^* T(Y \setminus \Sigma)) = \mathcal{H}^+ \oplus \mathcal{H}^-$ with grading induced by $\epsilon = i^{p(d-1) + \frac{s}{2}}$ and $\ast \text{Dom}(D) \subset \text{Dom}(D)$, $D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$.

$A = C(Y) \in \mathfrak{A} = C^{Lip}(Y)$. Clearly $\mathfrak{A}$ is closed under holomorphic calculus of $A$ and, by Lemma 2.1, leaves Dom($D$) invariant and $\|[D, a]\| < \infty$, $\forall a \in \mathfrak{A}$.

Proposition 2.10. $(\mathcal{H}, D)$ is an unbounded $p$-summable Fredholm module for $p > m$. Similar result holds for $X^\dagger$.

Proof. By [G], $\nabla = \partial \partial^* + \delta \delta^* = D^2$ is self-adjoint. By [C2, Lemma 7.1], $D^2$ has a discrete spectrum with finite multiplicities. By [C2, Theorem 7.2], we have the following asymptotic expansion of the heat kernel:

$$\text{Tr}(e^{-tD^2}) = \int_X \text{tr} e^{-tD^2}(x,x) \sim c_m + \sum_{j=0}^{m-2} c_j t^{-\frac{m-j}{2} + \frac{1}{2}} + O(t^{-\frac{1}{2}}).$$

i.e.

$$\int_X \text{tr} e^{-tD^2}(x,x) \sim t^{-\frac{m}{2}} c_0 + O(t^{-\frac{m-1}{2}}).$$

By Karamata Theorem [BGV, p.94],

$$N(\lambda) \sim \frac{c_0}{\Gamma(\frac{m+2}{2})} \lambda^{\frac{m}{2}}$$

$$\lambda_j \sim C j^{\frac{m}{2}}$$

$$\text{Tr}((1+D^2)^{-\frac{m}{2}}) < \infty \quad \text{for } p > m.$$ 

By Proposition 2.9 and Theorem 2.7, one can obtain the asymptotic expansion on $X^\dagger$. Then result for $X^\dagger$ follows as above.

3. “Straight” Chern character and $L$-class

3.1. “Straight” Chern character. Since we have a Fredholm module for the signature operator, we can repeat the construction of the “straight” Chern character as in [CM] and [MW1]. Let’s recall their construction:

Let $u$ be an even smooth function on $\mathbb{R}$ such that $\overline{v}(x) = 1 - x^2u(x)$ is Schwartz and both $\overline{\sigma}$ and $\sigma$ have Fourier transforms supported on $(-\frac{1}{2}, \frac{1}{2})$. As $\overline{\sigma}$ and $\sigma$ are even,

$$\overline{\sigma}(x) = u(x^2), \quad \sigma(x) = v(x^2)$$

for some smooth functions $u$ and $v$.

Clearly $v$ is also Schwartz and so is

$$w(x) = v(x)(1 + v(x))u(x).$$

Let $\gamma = \nu^\frac{m+1}{2}(-1)^{p(m-p)}i^\frac{p(p+1)}{2} * p : L_2(\Omega^p(X)) \rightarrow L_2(\Omega^{m-p}(X)).$
Then we consider the idempotent

\[ P(tD) = \left( \begin{array}{cc} (v(t^2D^2)^{2q}) & w(t^2D^2) \cdot tD \gamma \\ -v(t^2D^2) \cdot tD \gamma & (v(t^2D^2)^{2q}) \end{array} \right) + \left( \begin{array}{cc} \frac{1-\gamma}{2} & 0 \\ 0 & \frac{1-\gamma}{2} \end{array} \right) \]

and define an Alexander-Spanier cycle \( \Lambda_\ast(tD) \) as follows:

Let \( f^0, \ldots, f^{2q} \in C(X^\dagger) \),

\[ q = 0, \quad \Lambda_0(tD)(f^0) := \frac{1}{2} \text{Tr} \left( P(tD)f^0 - \left( \begin{array}{cc} \frac{1-\gamma}{2} & 0 \\ 0 & \frac{1-\gamma}{2} \end{array} \right) f^0 \right), \]

\[ q > 0, \quad \Lambda_{2q}(tD)(f^0 \otimes \cdots \otimes f^{2q}) := \frac{(2\pi i)^q}{q!(2q+1)2} \text{Tr} \left( \sum_{\sigma \in S_{2q+1}} \text{sgn}(\sigma) P(tD)f^{\sigma(0)} \cdots P(tD)f^{\sigma(2q)} \right), \]

and \( \overline{c_{2q}}(D) := \lim_{t \to 0} \Lambda_{2q}(tD) \).

By using the finite propagation speed, as in [MW1], we have

**Proposition 3.1.** a) Given two isometric open embedding of admissible Riemannian pseudomanifolds \((U, \rho_U) \hookrightarrow (Y_i, \rho_{Y_i}), \ i = 1, 2, \) and a compact subset \( K \) of \( U \), there is a \( \delta > 0 \) such that \( \forall t \in (0, \delta) \) and \( f^0, \ldots, f^k \in C^L_{cp}(U) \) with at least one of the \( f^j \)'s having support inside \( K \), one has

\[ \Lambda_k(P(tD_1))|_U(f^0 \otimes \cdots \otimes f^k) = \Lambda_k(P(tD_2))|_U(f^0 \otimes \cdots \otimes f^k), \]

where \( D_1 \) is the signature operator of \((Y_i, \rho_{Y_i})\) as defined in Section 2.1.

b) For any \( 2q > m = \dim Y \) and \( f^i \in C^L_{cp}(Y) \), one has

\[ \lim_{t \to 0} \Lambda_{2q}(tD)(f^0 \otimes \cdots \otimes f^{2q}) = 0. \]

Moreover, \( \Lambda_{2q}(tD) = \partial \int_0^1 \nu_{2q+1}(sD)ds \) where \( \nu \) is defined as in [MW1, Section 3.2].

c) The results in (a) and (b) hold for \( X^\dagger \).

**Proof.** Same as [MW1, Theorem 2.2, Theorem 3.3].

**Proposition 3.2.** Let \( \delta_L = \frac{1}{2}(1 - (-1)^\ell) \). For any \( f^i \in C^\infty_S(X^\dagger) \),

\[ \lim_{t \to 0} \Lambda_{2q}(tD)(f^0 \otimes \cdots \otimes f^{2q}) = \int_M 2\pi \mathcal{L}(R(g^M)) f^0 df^1 \wedge \cdots \wedge df^{2q} \]

\[ -\delta_L \eta(L) 2\pi \int_N \mathcal{L}(R(g^N)) f^0 df^1 \wedge \cdots \wedge df^{2q}, \]

where \( \mathcal{L}(R(g^M)) \) and \( \mathcal{L}(R(g^N)) \) are the Atiyah-Hirzebruch \( \mathcal{L} \)-polynomials in the curvature of the Levi-Civita connection of the metrics \( g^M \) and \( g^N \) respectively.

**Proof.** Following [MW2], we choose \( \rho_0 \in C^\infty([0,1]) \) with \( \rho_0(r) \in [0,1] \) such that

\[ \rho_0(r) = \begin{cases} 1 & \forall r \in [0, \frac{1}{2}], \\
0 & \forall r \in [\frac{3}{4}, 1], \end{cases} \]

and define \( \rho : X^\dagger \to \mathbb{R} \) by

\[ \rho(x) = \begin{cases} 0 & \text{if } x \in M, \\
\rho_0(r) & \text{if } x = (r, s, y) \in c(L) \times N. \end{cases} \]

Clearly \( \rho \in C^\infty_S(X^\dagger) \).
Let $f = f^0 \otimes \cdots \otimes f^{2q}$. Write $f = f^{(1)} + f^{(2)}$, where $f^{(1)} := (\rho f^0) \otimes f^1 \cdots \otimes f^{2q}$ and $f^{(2)} := ((1 - \rho)f^0) \otimes f^1 \cdots \otimes f^{2q}$.

(1) \[ \lim_{t \to 0} \langle A_{2q}(tD), f^{(2)} \rangle = \lim_{t \to 0} \langle A_{2q}(tD), f^{(1)} \rangle + \lim_{t \to 0} \langle A_{2q}(tD), f^{(2)} \rangle. \]

By locality (Proposition 3.1) and \[MW1, Theorem 3.4\],

\[ \lim_{t \to 0} \langle A_{2q}(tD), f^{(2)} \rangle = \frac{2}{\pi} \int_M \mathcal{L}(R(g^M))(1 - \rho)df^0 \wedge \cdots \wedge df^{2q} \]

\[ = \frac{2}{\pi} \int_M \mathcal{L}(R(g^M))df^0 \wedge \cdots \wedge df^{2q}. \]

Case 1: $\ell$ is even and $n$ is odd.

By locality (Proposition 3.1) and product formula \[MW2, Proposition 5.1\],

\[ \lim_{t \to 0} \langle A_{2q}(tD), f^{(1)} \rangle = 0. \]

Case 2: $\ell$ is odd and $n$ is even.

As $\ell$ is odd, by Thom's Theorem \[St, p.183\], there exists a smooth, compact and oriented manifold $W$ with $\partial W = 2L$.

Let $Y = (W \cup 2c(L)) \times N$, $M = 2M \cup (-W \times N)$, $\pi_Y : Y \to N$ be the projection map onto the second factor, and $f_Y = \pi_Y^*(f^0 \otimes \cdots \otimes f^{2q}) \in C^\infty_S(Y)$.

Let $(f)_2$ be the function on $2X$ which equals $f$ on each copy of $X$. Now $(f^{(1)})_2 = f_Y = f_Y^{(1)} - f_Y^{(2)}$. By locality (Proposition 3.1),

\[ 2\langle A_{2q}(tD_X), f^{(1)} \rangle = \langle A_{2q}(tD_X), (f^{(1)})_2 \rangle \]

\[ = \langle A_{2q}(tD_Y), f_Y^{(1)} \rangle \]

\[ = \langle A_{2q}(tD_Y), f_Y \rangle - \langle A_{2q}(tD_Y), f_Y^{(2)} \rangle. \]

By using the index theorem in \[C2\] instead of the McKean-Singer formula at the end of the proof in \[MW2, Lemma 5.2\], we see that we still have a product formula for space with conical singularities. Therefore,

\[ \langle A_{2q}(tD_Y), f_Y \rangle = \text{sign}(W \cup 2c(L)) \cdot \langle A_{2q}(tD_N), f^0 \otimes \cdots \otimes f^{2q} \rangle \]

\[ = \text{sign}(W \cup 2c(L)) 2 \frac{2}{\pi} \int_N \mathcal{L}(R(g^N))df^0 \wedge \cdots \wedge df^{2q}. \]

The last line follows from \[MW1, Theorem 3.4\]. Now,

\[ 2\langle A_{2q}(tD_X), f^{(2)} \rangle - \langle A_{2q}(tD_Y), f_Y^{(2)} \rangle = \langle A_{2q}(tD_A), f_A^{(2)} \rangle, \]

where $f_A^{(2)} = (f^{(2)})_2 \sqcup f_Y^{(2)}$ and $A = 2X \cup -Y = \tilde{M} \cup 2(c(L) \cup (-c(L))) \times N$.

Let $Z = (c(L) \cup (-c(L))) \times N$. Then $f_Z^{(2)} = \tilde{f} \sqcup f_Z^{(2)}$.

Clearly, $\langle A_{2q}(tD_Z), f_Z^{(2)} \rangle = 0$. Then by locality (Proposition 3.1),

\[ 2\langle A_{2q}(tD_X), f^{(2)} \rangle - \langle A_{2q}(tD_Y), f_Y^{(2)} \rangle = \langle A_{2q}(tD_A), \tilde{f} \rangle. \]
Together with (1), (2), (3) and (4), we have
\[
2\lim_{t \to 0} \Lambda_{2q}(tD, \tilde{f}) = \lim_{t \to 0} \left( \langle \Lambda_{2q}(tD_{Y}), f_Y \rangle - \langle \Lambda_{2q}(tD_{Y}), f_Y^{(2)} \rangle + 2\langle \Lambda_{2q}(tD_{X}), f_Y^{(2)} \rangle \right)
\]
\[
= \text{sign}(W \cup 2c(L)) \, 2^{n_2} \int_{N} L(R(g^N)) f^0 df^1 \cdots df^{2q} + \lim_{t \to 0} \langle \Lambda_{2q}(tD_{\tilde{M}}), \tilde{f} \rangle.
\]
As $\tilde{M}$ is smooth, by [MW1, Theorem 3.4] and the product formula [MW1, Proposition 5.1], we have
\[
\lim_{t \to 0} \langle \Lambda_{2q}(tD_{\tilde{M}}), \tilde{f} \rangle = 2 \int_{M} 2^{n_2} L(R(g^M)) f^0 df^1 \cdots df^{2q}
\]
\[
- \int_{W} 2^{\frac{q+1}{2}} L(R(g^W)) \cdot \int_{N} 2^{n_2} L(R(g^N)) f^0 df^1 \cdots df^{2q}.
\]
But by [C2],
\[
\text{sign}(W \cup 2c(L)) = \int_{W} 2^{\frac{q+1}{2}} L(R(g^W)) - 2\eta(L).
\]
Hence the result follows. 

\[ \square \]

3.2. Analytic realization of Goresky-MacPherson-Siegel $L$-class. In this subsection, we will show that, under a condition on the link, the “straight” Chern character and the Goresky-MacPherson-Siegel $L$-class are the same (up to constants). This gives an analytic realization of $L$-class.

In the following, we will denote the $L$-class as an element in $H_k(X^1)$ and $\overline{H}_k(X^1)$ by the same symbol.

**Theorem 3.3.** If the space $X^1$ is even dimensional and $2L$ is oriented cobordant to zero, then
\[
[\overline{ch}_{2q}(D)] = [2^q L_{2q}(X^1)].
\]

**Proof.** Notice that any element in $[X^1, S^{2q}]$ is homotopic to an element in
\[
C^\infty_S(X^1, S^{2q}) = \{ f \in C(X^1, S^{2q}) : \text{smooth on } X \text{ such that } f|_{c(L) \times N} = pr^*(g) \text{ for } g \in C^\infty(N, S^{2q}) \}.
\]

By [GM], it suffices to show that for any $f \in C^\infty_S(X^1, S^{2q})$ which is transverse,
\[
\langle \overline{ch}_{2q}(D), f^*(s^{2q}) \rangle = 2^q \sigma(f^{-1}(p)),
\]
where $\sigma$ is the signature in Section 1.3 and $s^{2q}$ is a fixed generator of $\overline{H}^k(S^{2q})$. By [BT, p.37], $s^{2q}$ and $f^*(s^{2q})$ are sums of elementary tensors. Note that the normalization constants of $L$-classes are different from [MS].

Then $f|_{c(L) \times N} = (pr)^*(g)$ for some $g : N \to S^{2q}$ which is smooth and transverse at $p$. In the rest of this proof, we will adopt the notation used in the proof of Proposition 3.2.
As in the proof of Proposition 3.2, we have
\[
\left< \overline{\mathcal{H}_2}(D), f^*(s^2q) \right>
\]
\[
= \frac{1}{2} \left( \text{sign}(W \cup 2c(L)) \left< \overline{\mathcal{H}_2}(D_N), g^*(s^2q) \right> + \left< \overline{\mathcal{H}_2}(D_N), \tilde{f}^*(s^2q) \right> \right)
\]
\[
= \frac{2^n}{2} \left[ \sigma(W \cup 2c(L)) \sigma(g^{-1}(p)) + \sigma(2(f^{-1}(p) \cap M) \cup (-W \times g^{-1}(p))) \right]
\]
\[
= \frac{2^n}{2} \left[ \sigma(W \cup 2c(L)) \sigma(g^{-1}(p)) + \sigma(2(f^{-1}(p) \cap M) \cup (-W \times g^{-1}(p))) \right]
\]
\[
= 2^n \sigma((c(L) \times g^{-1}(p)) \cup (f^{-1}(p) \cap M))
\]
\[
= 2^n \sigma(f^{-1}(p)).
\]
The last three lines follow from the multiplicativity and additivity of \( \sigma \) [Si].

\[\square\]

**Remark 3.4.** 1. If the space \( X^\dagger \) is even dimensional, then \( \left[ \mathcal{L}_{2q+1}(X^\dagger) \right]\) = \( \{0\} \).

2. By Thom’s Theorem [St, p. 183], \( 2L \) is oriented cobordant to zero iff all Pontryagin numbers are zero. It is always true if \( \ell \not\equiv 0 \pmod{4} \).

**Theorem 3.5.** For any admissible pseudomanifold \( X^\dagger \) with one singular stratum such that \( 2L \) is oriented cobordant to zero, one has
\[
L_\ast(X^\dagger; \rho_X) := 2^{2m} \mathcal{L}_m(R(g^M)) \oplus -\delta_\ell \eta(L) 2^{2n'} \mathcal{L}_{n'}(R(g^N)) \in \Omega_\ast(X^\dagger)_{SA}
\]
is a cycle which represents the \( \mathcal{L} \)-class of \( X^\dagger \), where \( m' = \frac{m-n}{4}, n' = \frac{n-\delta_\ell}{4}, \delta_\ell = \frac{1}{2}(1 - (-1)^\ell) \) and \( \mathcal{L}_k(R(\cdot)) := 0 \) for \( k' \not\in \mathbb{Z} \).

**Proof.** We will divide this into two cases.

Case 1: \( \dim X \) is even.

It follows from the previous theorem (cf. [MW2, Theorem 4.3]).

Case 2: \( \dim X \) is odd.

Let \( q_1, q_2 \) be the projections onto the first and second factors of \( X^\dagger \times S^1 \) respectively. Then by definition,
\[
\mathcal{L}_k(X^\dagger)(\omega) = \mathcal{L}_{k+1}(X^\dagger \times S^1)(q_1^* \omega \wedge q_2^* s^1),
\]
where \( \omega \in H^k(X^\dagger) \) and \( H^1(S^1) = \langle s^1 \rangle \).

The result follows from the definition of the pairing of Sullivan complexes (Section 1.2) and Case 1. \( \square \)

4. **Index theorem**

In this section, we will establish the index theorem for the pseudomanifolds with one singular stratum. And then we will identify the “straight” Chern character with the K-homology Chern character.

**Theorem 4.1.** If the space \( X^\dagger \) is even dimensional, then
\[
\text{Ind}(D_E) = 2^n \int_M \mathcal{L}(R(g^M)) \wedge ch(E) - \delta_\ell \eta(L) 2^n \int_N \mathcal{L}(R(g^N)) \wedge ch(E).
\]

**Proof.** By Proposition 2.9 and Theorem 2.7, we have
\[
\int_{c(L) \times N} \text{tr}_0 e^{-\epsilon D_k^2}(x, x) = \int_{c(L) \times N} \text{tr}_0 e^{-\epsilon D_k^2}(x, x) + O(t),
\]
where \( D_E \) is the twisted signature operator on \( c_{0, \infty}(L) \times N \).
Hence, $\text{Ind}(D_E) = \lim_{t \to 0} \left( \int_M \text{tr}_x e^{-tD_E^2(x,x)} + \int_{\gamma(L) \times N} \text{tr}_x e^{-tD_N^2(x,x)} \right)$.

1° $\ell$ is odd and $n$ is even.

Notice that $\tilde{D}_E = D_{e_0 \infty} \otimes I + I \otimes D_{N,E}$. Therefore, $\text{tr}_x e^{-tD_E^2(x,x)} = \text{tr}_x e^{-tD_{e_0 \infty}^2(x,x)} \text{tr}_x e^{-tD_{N,E}(x,x)}$. By [C2],

$$\int_{\gamma(L) \times N} \text{tr}_x e^{-tD_E^2(x,x)} = -\eta(L) \int_N 2\pi \mathcal{L}(R(g^N)) \wedge \text{ch}(E).$$

2° $\ell$ is even and $n$ is odd.

Note that $\tilde{D} = D_{e_0 \infty} \otimes I + \phi \otimes D_{N,E}$ where $\phi = (-1)^k$ on $L^2(\Omega^k(c(L)))$. Also, $\gamma = \frac{1}{i} \gamma_1 \gamma_2, \gamma_1 = \frac{i}{1 + i} c(\text{vol}_{c(L)}) \otimes I$ and $\gamma_2 = \frac{i}{1 - i} \phi \otimes c(\text{vol}_N)$ where $c(\text{vol}_{c(L)})$ and $c(\text{vol}_N)$ are Clifford multiplication by volume elements in $c(L)$ and $N$ respectively. Then,

$$\text{tr}_x e^{-tD_E^2(x,x)} = \frac{1}{i} \text{tr}(\gamma_1 \gamma_2 e^{-tD_{e_0 \infty}^2(x,x)})(x,x)$$

$$= \frac{1}{i} \text{tr}(\gamma_1 e^{-tD_{e_0 \infty}^2(x,x)})(x,x)$$

$$= \frac{1}{i} \text{tr}(\gamma_2 \gamma_1 e^{-tD_{e_0 \infty}^2(x,x)})(x,x)$$

$$= \frac{1}{i} \text{tr}(-\gamma_1 \gamma_2 e^{-tD_{e_0 \infty}^2(x,x)})(x,x)$$

Hence the result follows. \hfill \Box

**Remark 4.2.** The above theorem is a special case of the index theorem announced in [BC]. They considered the index theory of twisted signature operators for fibration with fibers having conical singularities.

**Corollary 4.3.** If the space $X^\dagger$ is even dimensional, then

$$[\text{ch}(D)] = [\bar{\text{ch}}(D)].$$

**Proof.** By [MS, p.196 and Theorem 9.1], one can see that

$$\text{ch}(E_X) = \text{ch}(E_M) \oplus \text{ch}(E_N) \in H^*(\Omega^*(X^\dagger),SA).$$

Thus, by Theorem 4.1, $\text{Ind}(D_E) = \langle 2\pi L_s(X^\dagger, \rho_{X^\dagger}), \text{ch}(E) \rangle$. Hence the result follows from Theorem 3.3. \hfill \Box

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**References**


[CM] A. Connes, H. Moscovici, Cyclic cohomology, the Novikov conjecture, and hyperbolic groups, Topology 29 No.3 (1990), 345-388. MR 92a:58137


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