APPLICATIONS OF PSEUDO-MONOTONE OPERATORS
WITH SOME KIND OF UPPER SEMICONTINUITY
IN GENERALIZED QUASI-VARIATIONAL INEQUALITIES
ON NON-COMPACT SETS

MOHAMMAD S. R. CHOWDHURY AND KOK-KEONG TAN

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Abstract. Let $E$ be a topological vector space and $X$ be a non-empty subset of $E$. Let $S : X \to 2^X$ and $T : X \to 2^{E^*}$ be two maps. Then the generalized quasi-variational inequality (GQVI) problem is to find a point $\hat{y} \in S(\hat{y})$ and a point $\hat{w} \in T(\hat{y})$ such that $\text{Re} \langle \hat{w}, \hat{y} - x \rangle \leq 0$ for all $x \in S(\hat{y})$. We shall use Chowdhury and Tan’s 1996 generalized version of Ky Fan’s minimax inequality as a tool to obtain some general theorems on solutions of the GQVI on a paracompact set $X$ in a Hausdorff locally convex space where the set-valued operator $T$ is either strongly pseudo-monotone or pseudo-monotone and is upper semicontinuous from $\text{co}(A)$ to the weak-$*$-topology on $E^*$ for each non-empty finite subset $A$ of $X$.

1. Introduction

If $X$ is a set, we shall denote by $2^X$ the family of all non-empty subsets of $X$ and by $\mathcal{F}(X)$ the family of all non-empty finite subsets of $X$. Let $E$ be a topological vector space. We shall denote by $E^*$ the continuous dual of $E$, by $\langle w, x \rangle$ the pairing between $E^*$ and $E$ for $w \in E^*$ and $x \in E$ and by $\text{Re} \langle w, x \rangle$ the real part of $\langle w, x \rangle$. If $X \subset E$, $S : X \to 2^X$ and $T : X \to E^*$, the quasi-variational inequality problem (QVI) is to find a point $\hat{y} \in S(\hat{y})$ such that $\text{Re} \langle T(\hat{y}), \hat{y} - x \rangle \leq 0$ for all $x \in S(\hat{y})$. The QVI was first introduced by Bensoussan and Lions in 1973 (see, e.g., [2]) in connection with impulse control. Again, if we consider a set-valued map $T : X \to 2^{E^*}$, then the generalized quasi-variational inequality problem (GQVI) is to find a point $\hat{y} \in S(\hat{y})$ and a point $\hat{w} \in T(\hat{y})$ such that $\text{Re} \langle \hat{w}, \hat{y} - x \rangle \leq 0$ for all $x \in S(\hat{y})$. The GQVI was introduced by Chan and Pang [4] in 1982 if $E = \mathbb{R}^n$ and by Shih and Tan [11] in 1985 if $E$ is infinite dimensional.

In this paper, we shall use Chowdhury and Tan’s generalized version [5, Theorem 2] of Ky Fan’s minimax inequality [8, Theorem 1] as a tool to obtain some

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general theorems on solutions of the GQVI on a paracompact set \( X \) in a locally convex Hausdorff topological vector space where the set-valued operator \( T \) is strongly pseudo-monotone or pseudo-monotone and is upper semicontinuous from \( co(A) \) to the weak*-topology on \( E^* \) for each \( A \in \mathcal{F}(X) \).

We shall use our following set-valued generalization of the classical pseudo-monotone operator. The classical definition of a pseudo-monotone operator was introduced by Brézis, Nirenberg and Stampacchia in [3]. For a slightly general definition of a pseudo-monotone operator we refer to [5, Definition 1].

**Definition 1.1.** Let \( E \) be a topological vector space, \( X \) be a non-empty subset of \( E \) and \( T : X \rightarrow 2^{E^*} \). If \( h : X \rightarrow \mathbb{R} \), then \( T \) is said to be (1) \( h \)-pseudo-monotone if for each \( y \in X \) and every net \( \{y_\alpha\}_{\alpha \in \Gamma} \) in \( X \) converging to \( y \) with

\[
\limsup_{\alpha}[\inf_{u \in T(y_\alpha)} \text{Re}(u, y_\alpha - y) + h(y_\alpha) - h(y)] \leq 0,
\]

we have

\[
\liminf_{\alpha}[\inf_{u \in T(y_\alpha)} \text{Re}(u, y_\alpha - x) + h(y_\alpha) - h(x)] \geq \inf_{w \in T(y)} \text{Re}(w, y - x) + h(y) - h(x)
\]

for all \( x \in X \); (2) pseudo-monotone if \( T \) is \( h \)-pseudo-monotone with \( h \equiv 0 \).

2. Generalized quasi-variational inequalities for strongly pseudo-monotone operators

In this section we shall introduce the notion of strongly pseudo-monotone operators and obtain some general theorems on solutions of the GQVI on paracompact sets in locally convex Hausdorff topological vector spaces.

We shall begin with the following:

**Definition 2.1.** Let \( E \) be a topological vector space, \( X \) be a non-empty subset of \( E \) and \( T : X \rightarrow 2^{E^*} \). If \( h : X \rightarrow \mathbb{R} \), then \( T \) is said to be (1) strongly \( h \)-pseudo-monotone if for each continuous function \( \theta : X \rightarrow [0, 1] \), for each \( y \in X \) and every net \( \{y_\alpha\}_{\alpha \in \Gamma} \) in \( X \) converging to \( y \) with

\[
\limsup_{\alpha}[\theta(y_\alpha)(\inf_{u \in T(y_\alpha)} \text{Re}(u, y_\alpha - y) + h(y_\alpha) - h(y))] \leq 0
\]

we have

\[
\limsup_{\alpha}[\theta(y_\alpha)(\inf_{u \in T(y_\alpha)} \text{Re}(u, y_\alpha - x) + h(y_\alpha) - h(x))] \\
\geq [\theta(y)(\inf_{w \in T(y)} \text{Re}(w, y - x) + h(y) - h(x))]
\]

for all \( x \in X \); (2) strongly pseudo-monotone if \( T \) is strongly \( h \)-pseudo-monotone with \( h \equiv 0 \).

Clearly, every strongly pseudo-monotone operator is also a pseudo-monotone operator as defined in [5].

**Proposition 2.1.** Let \( X \) be a non-empty subset of a topological vector space \( E \). If \( T : X \rightarrow E^* \) is monotone and continuous from the relative weak topology on \( X \) to the weak* topology on \( E^* \), then \( T \) is strongly pseudo-monotone.

**Proof.** Let us consider any arbitrary continuous function \( \theta : X \rightarrow [0, 1] \). Suppose \( \{y_\alpha\}_{\alpha \in \Gamma} \) is a net in \( X \) and \( y \in X \) with \( y_\alpha \rightarrow y \) (and

\[
\limsup_{\alpha}[\theta(y_\alpha)(\text{Re}(Ty_\alpha, y_\alpha - y))] \leq 0.
\]
Then for any $x \in X$ and $\epsilon > 0$, there are $\beta_1, \beta_2 \in \Gamma$ with $|\theta(y_\alpha) \Re (Ty_\alpha, y_\alpha - y)| < \frac{\epsilon}{2}$ for all $\alpha \geq \beta_1$ and $|\theta(y_\alpha) \Re (Ty_\alpha, Ty_\alpha - y - x)| < \frac{\epsilon}{2}$ for all $\alpha \geq \beta_2$. Choose $\beta_0 \in \Gamma$ with $\beta_0 \geq \beta_1, \beta_2$. Thus

$$\theta(y_\alpha) \Re (Ty_\alpha, y_\alpha - x) = \theta(y_\alpha) \Re (Ty_\alpha, y_\alpha - y) + \theta(y_\alpha) \Re (Ty_\alpha, y - x)$$
$$\geq \theta(y_\alpha) \Re (Ty, y_\alpha - y) + \theta(y_\alpha) \Re (Ty_\alpha, y - x)$$
$$= \theta(y_\alpha) \Re (Ty, y_\alpha - y) + \theta(y_\alpha) \Re (Ty_\alpha - Ty, y - x) + \theta(y_\alpha) \Re (Ty, y - x)$$
$$> \frac{\epsilon}{2} - \epsilon + \frac{\epsilon}{2} + \theta(y_\alpha) \Re (Ty, y - x)$$

so that $\inf_{\alpha \geq \beta_0} \theta(y_\alpha) \Re (Ty_\alpha, y_\alpha - x) \geq -\epsilon + \inf_{\alpha \geq \beta_0} \theta(y_\alpha) \Re (Ty, y - x)$. It follows that $\limsup_{\beta} \theta(y_\beta) \Re (Ty_\beta, y_\beta - x) \geq \liminf_{\beta} \theta(y_\beta) \Re (Ty_\beta, y_\beta - x) \geq -\epsilon + \theta(\gamma) \Re (Ty, y - x)$. As $\epsilon > 0$ is arbitrary,

$$\limsup_{\beta} \theta(y_\beta) \Re (Ty_\beta, y_\beta - x) \geq \theta(\gamma) \Re (Ty, y - x).$$

Hence $T$ is strongly pseudo-monotone.

We shall now establish the following result:

**Theorem 2.1.** Let $E$ be a locally convex Hausdorff topological vector space, $X$ be a non-empty paracompact convex subset of $E$ and $h : E \to \mathbb{R}$ be convex. Let $S : X \to 2^X$ be upper semicontinuous such that each $S(x)$ is compact convex and $T : X \to 2^{E^*}$ be strongly $h$-pseudo-monotone and be upper semicontinuous from $co(A)$ to the weak* topology on $E^*$ for each $A \in F(X)$ such that each $T(x)$ is weak*-compact convex. Suppose that the set

$$\Sigma = \{ y \in X : \sup_{x \in S(y)} \inf_{w \in T(y)} \Re (w, y - x) + h(y) - h(x) > 0 \}$$

is open in $X$. Suppose further that there exist a non-empty compact subset $K$ of $X$ and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and $\inf_{w \in T(y)} \Re (w, y - x_0) + h(y) - h(x_0) > 0$ for all $y \in X \setminus K$. Then there exists $\bar{y} \in K$ such that (i) $\bar{y} \in S(\bar{y})$ and (ii) there exists $\bar{w} \in T(\bar{y})$ with $\Re (\bar{w}, \bar{y} - x) \leq h(x) - h(\bar{y})$ for all $x \in S(y)$.

**Proof.** We divide the proof into two steps:

**Step 1.** There exists a point $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$ and

$$\sup_{x \in S(\hat{y})} \inf_{w \in T(\hat{y})} \Re (w, \hat{y} - x) + h(\hat{y}) - h(x) \leq 0.$$

Suppose the contrary. Then for each $y \in X$, either $y \notin S(y)$ or there exists $x \in S(y)$ such that $\inf_{w \in T(y)} \Re (w, y - x) + h(y) - h(x) > 0$; that is, $y \notin S(y)$ or $y \in \Sigma$. If $y \notin S(y)$, then by Hahn-Banach separation theorem, there exists $p \in E^*$ such that $\Re (p, y) - \sup_{y \in S(y)} \Re (p, x) > 0$. For each $y \in X$, set $\gamma(y) := \sup_{x \in S(y)} \inf_{w \in T(y)} \Re (w, y - x) + h(y) - h(x)$. Let $V_0 := \{ y \in X : \gamma(y) > 0 \} = \Sigma$ and for each $p \in E^*$, set $V_p := \{ y \in X : \Re (p, y) - \sup_{x \in S(y)} \Re (p, x) > 0 \}$.

Then $X = V_0 \cup \bigcup_{p \in E} V_p$. Since each $V_p$ is open in $X$ by Lemma 1 in [11] and $V_0$ is open in $X$ by hypothesis, $\{ V_0, V_p : p \in E^* \}$ is an open covering for $X$. Since $X$ is paracompact, there is a continuous partition of unity $\{ \beta_0, \beta_p : p \in E^* \}$ for $X$ subordinated to the open cover $\{ V_0, V_p : p \in E^* \}$ (see, e.g., Theorem VIII.4.2 of Dugundji in [7]); that is, for each $p \in E^*$, $\beta_p : X \to [0, 1]$ and $\beta_0 : X \to [0, 1]$ are continuous functions such that for each $p \in E^*$, $\beta_p(y) = 0$ for all $y \in X \setminus V_p$ and $\beta_0(y) = 0$ for all $y \in X \setminus V_0$ and $\{ \supp \beta_0, \supp \beta_p : p \in E^* \}$ is locally finite and $\beta_0(y) + \sum_{p \in E^*} \beta_p(y) = 1$ for each $y \in X$. Note that for each $A \in F(X)$, $h$ is

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continuous on \( co(A) \) (see e.g. [10, Corollary 10.1.1, p.83]). Define \( \phi : X \times X \to \mathbb{R} \) by

\[
\phi(x, y) = \beta_0(y) \left[ \min_{w \in T(y)} Re(w, y - x) + h(y) - h(x) \right] + \sum_{p \in E^*} \beta_p(y) Re(p, y - x)
\]

for each \( x, y \in X \). Then we have the following.

(1) Since \( E \) is Hausdorff, for each \( A \in F(X) \) and each fixed \( x \in co(A) \), the map \( y \mapsto \min_{w \in T(y)} Re(w, y - x) + h(y) - h(x) \) is lower semicontinuous on \( co(A) \) by Lemma 3 in [5] and the fact that \( h \) is continuous on \( co(A) \) and therefore the map \( y \mapsto \beta_0(y) [ \min_{w \in T(y)} Re(w, y - x) + h(y) - h(x) ] \) is lower semicontinuous on \( co(A) \) by Lemma 3 in [12]. Also for each fixed \( x \in X \), \( y \mapsto \sum_{p \in E^*} \beta_p(y) Re(p, y - x) \) is continuous on \( X \). Hence, for each \( A \in F(X) \) and each fixed \( x \in co(A) \), the map \( y \mapsto \phi(x, y) \) is lower semicontinuous on \( co(A) \).

(2) For each \( A \in F(X) \) and for each \( y \in co(A) \), \( \min_{x \in A} \phi(x, y) \leq 0 \). Indeed, if this were false, then for some \( A = \{ x_1, \cdots, x_n \} \in F(X) \) and some \( y \in co(A) \) (say \( y = \sum_{i=1}^n \lambda_i x_i \) where \( \lambda_1, \cdots, \lambda_n \geq 0 \) with \( \sum_{i=1}^n \lambda_i = 1 \), we have \( \min_{1 \leq i \leq n} \phi(x_i, y) > 0 \). Then for each \( i = 1, \cdots, n \),

\[
\beta_0(y) \left[ \min_{w \in T(y)} Re(w, y - x_i) + h(y) - h(x_i) \right] + \sum_{p \in E^*} \beta_p(y) Re(p, y - x_i) > 0
\]

so that

\[
0 = \phi(y, y) = \beta_0(y) \left[ \min_{w \in T(y)} Re(w, y - \sum_{i=1}^n \lambda_i x_i) + h(y) - h(\sum_{i=1}^n \lambda_i x_i) \right] + \sum_{p \in E^*} \beta_p(y) Re(p, y - \sum_{i=1}^n \lambda_i x_i)
\]

\[
\geq \sum_{i=1}^n \lambda_i \beta_0(y) \left[ \min_{w \in T(y)} Re(w, y - x_i) + h(y) - h(x_i) \right] + \sum_{p \in E^*} \beta_p(y) Re(p, y - x_i) > 0,
\]

which is a contradiction.

(3) Suppose \( A \in F(X) \), \( x, y \in co(A) \) and \( \{ y_\alpha \}_{\alpha \in \Gamma} \) is a net in \( X \) converging to \( y \) with \( \phi(tx + (1 - t)y, y_\alpha) \leq 0 \) for all \( \alpha \in \Gamma \) and all \( t \in [0,1] \).

Then for \( t = 0 \) we have \( \phi(y, y_\alpha) \leq 0 \) for all \( \alpha \in \Gamma \), i.e.,

\[
\beta_0(y_\alpha) \left[ \min_{w \in T(y_\alpha)} Re(w, y_\alpha - y) + h(y_\alpha) - h(y) \right] + \sum_{p \in E^*} \beta_p(y_\alpha) Re(p, y_\alpha - y) \leq 0
\]

for all \( \alpha \in \Gamma \). Hence

\[
\limsup_{\alpha} [\beta_0(y_\alpha) \left( \min_{w \in T(y_\alpha)} Re(w, y_\alpha - y) + h(y_\alpha) - h(y) \right)] + \liminf_{\alpha} \left( \sum_{p \in E^*} \beta_p(y_\alpha) Re(p, y_\alpha - y) \right)
\]

\[
\leq \limsup_{\alpha} [\beta_0(y_\alpha) \left( \min_{w \in T(y_\alpha)} Re(w, y_\alpha - y) + h(y_\alpha) - h(y) \right)] + \sum_{p \in E^*} \beta_p(y_\alpha) Re(p, y_\alpha - y) \leq 0.
\]
Thus by (2.1) and (2.2), we have
\[ h \text{ is strongly } \alpha \text{ for all } y. \]
Consequently, whenever \( \beta \phi \), we have
\[ \limsup_{T(y)} [\beta_0(y) (\min_{w \in T(y)} Re\langle w, y - x \rangle + h(y) - h(x))] \leq 0. \]
Therefore \( \limsup_{T(y)} [\beta_0(y) (\min_{w \in T(y)} Re\langle w, y - x \rangle + h(y) - h(x))] \leq 0. \) Since \( T \) is strongly \( h \)-pseudo-monotone, we have
\[ \limsup_{T(y)} [\beta_0(y) (\min_{w \in T(y)} Re\langle w, y - x \rangle + h(y) - h(x))] \geq \beta_0(y) (\min_{w \in T(y)} Re\langle w, y - x \rangle + h(y) - h(x)). \]
Thus
\[ \limsup_{T(y)} [\beta_0(y) (\min_{w \in T(y)} Re\langle w, y - x \rangle + h(y) - h(x))] + \sum_{p \in E^*} \beta_p(y) Re\langle p, y - x \rangle \geq \beta_0(y) (\min_{w \in T(y)} Re\langle w, y - x \rangle + h(y) - h(x)) + \sum_{p \in E^*} \beta_p(y) Re\langle p, y - x \rangle. \]
For \( t = 1 \) we have \( \phi(x, y) \leq 0 \) for all \( \alpha \in \Gamma \), i.e.,
\[ \beta_0(y) [\min_{w \in T(y)} Re\langle w, y - x \rangle + h(y) - h(x)] + \sum_{p \in E^*} \beta_p(y) Re\langle p, y - x \rangle \leq 0 \]
for all \( \alpha \in \Gamma \). Therefore
\[ \limsup_{T(y)} [\beta_0(y) (\min_{w \in T(y)} Re\langle w, y - x \rangle + h(y) - h(x))] + \liminf_{\alpha} \sum_{p \in E^*} \beta_p(y) Re\langle p, y - x \rangle \leq \limsup_{T(y)} [\beta_0(y) (\min_{w \in T(y)} Re\langle w, y - x \rangle + h(y) - h(x))] + \sum_{p \in E^*} \beta_p(y) Re\langle p, y - x \rangle \leq 0. \]
Thus
\[ \limsup_{T(y)} [\beta_0(y) (\min_{w \in T(y)} Re\langle w, y - x \rangle + h(y) - h(x))] + \sum_{p \in E^*} \beta_p(y) Re\langle p, y - x \rangle \leq 0. \]
Hence by (2.1) and (2.2), we have \( \phi(x, y) \leq 0. \)
(4) By hypothesis, there exist a non-empty compact (and therefore closed) subset \( K \) of \( X \) and a point \( x_0 \in X \) such that \( x_0 \in K \cap S(y) \) and \( \inf_{w \in T(y)} Re\langle w, y - x_0 \rangle + h(y) - h(x_0) > 0 \) for each \( y \in X \setminus K \). Thus for each \( y \in X \setminus K \),
\[ \beta_0(y) [\inf_{w \in T(y)} Re\langle w, y - x_0 \rangle + h(y) - h(x_0)] > 0 \]
whenever \( \beta_0(y) > 0 \) and \( Re\langle p, y - x_0 \rangle > 0 \) whenever \( \beta_p(y) > 0 \) for \( p \in E^* \). Consequently,
\[ \phi(x_0, y) = \beta_0(y) [\inf_{w \in T(y)} Re\langle w, y - x_0 \rangle + h(y) - h(x_0)] + \sum_{p \in E^*} \beta_p(y) Re\langle p, y - x_0 \rangle > 0 \]
for all \( y \in X \setminus K \).
Then $\phi$ satisfies all hypotheses of Theorem 2 in [5]. Hence by Theorem 2 in [5], there exists a point $\hat{y} \in K$ such that $\phi(x, \hat{y}) \leq 0$ for all $x \in X$; i.e.,

$$\beta_0(\hat{y}) \left[ \inf_{w \in T(\hat{y})} Re\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x) \right] + \sum_{p \in E^*} \beta_p(\hat{y}) Re\langle p, \hat{y} - x \rangle \leq 0$$

for all $x \in X$.

If $\gamma(\hat{y}) = 0$, choose any $\hat{x} \in S(\hat{y})$; if $\gamma(\hat{y}) > 0$, choose any $\hat{x} \in S(\hat{y})$ such that

$$\inf_{w \in T(\hat{y})} Re\langle w, \hat{y} - \hat{x} \rangle + h(\hat{y}) - h(\hat{x}) \geq \frac{\gamma(\hat{y})}{2} > 0.$$ 

If $\beta_0(\hat{y}) > 0$, then $\hat{y} \in V_0 = \Sigma$ so that $\gamma(\hat{y}) > 0$; it follows that

$$\beta_0(\hat{y}) \left[ \inf_{w \in T(\hat{y})} Re\langle w, \hat{y} - \hat{x} \rangle + h(\hat{y}) - h(\hat{x}) \right] > 0.$$ 

If $\beta_p(\hat{y}) > 0$ for some $p \in E^*$, then $\hat{y} \in V_p$ and hence $Re\langle p, \hat{y} \rangle > \sup_{y \in S(\hat{y})} Re\langle p, y \rangle$ 

$$\geq Re\langle p, \hat{y} - \hat{x} \rangle > 0.$$ 

Then note that $\beta_p(\hat{y}) Re\langle p, \hat{y} - \hat{x} \rangle > 0$ whenever $\beta_p(\hat{y}) > 0$ for $p \in E^*$.

Since $\beta_0(\hat{y}) > 0$ or $\beta_p(\hat{y}) > 0$ for some $p \in E^*$, it follows that

$$\phi(\hat{x}, \hat{y}) = \beta_0(\hat{y}) \left[ \inf_{w \in T(\hat{y})} Re\langle w, \hat{y} - \hat{x} \rangle + h(\hat{y}) - h(\hat{x}) \right] + \sum_{p \in E^*} \beta_p(\hat{y}) Re\langle p, \hat{y} - \hat{x} \rangle > 0,$$

which contradicts (2.3). This contradiction proves Step 1.

**Step 2.** There exists a point $\hat{w} \in T(\hat{y})$ such that $Re\langle \hat{w}, \hat{y} - x \rangle + h(\hat{y}) - h(x) \leq 0$ for all $x \in S(\hat{y})$.

Note that for each fixed $x \in S(\hat{y})$, $w \mapsto Re\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x)$ is convex and continuous on $T(\hat{y})$ and for each fixed $w \in T(\hat{y})$, $x \mapsto Re\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x)$ is concave on $S(\hat{y})$. Thus by Kneser’s Minimax Theorem in [9] (see also Aubin [1, pp.40-41]), we have

$$\min_{w \in T(\hat{y})} \max_{x \in S(\hat{y})} [Re\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x)] = \max_{x \in S(\hat{y})} \min_{w \in T(\hat{y})} [Re\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x)].$$

Hence $\min_{w \in T(\hat{y})} \max_{x \in S(\hat{y})} [Re\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x)] \leq 0$ by Step 1. Since $T(\hat{y})$ is compact, there exists $\hat{w} \in T(\hat{y})$ such that $Re\langle \hat{w}, \hat{y} - x \rangle + h(\hat{y}) - h(x) \leq 0$ for all $x \in S(\hat{y})$.

If $X$ is compact, we obtain the following immediate consequence of Theorem 2.1:

**Theorem 2.2.** Let $E$ be a locally convex Hausdorff topological vector space, $X$ be a non-empty compact convex subset of $E$ and $h : E \to \mathbb{R}$ be convex. Let $S : X \to 2^X$ be upper semicontinuous such that each $S(x)$ is closed convex and $T : X \to 2^{E^*}$ be strongly $\rho$-pseudo-monotone and be upper semicontinuous from $co(A)$ to the weak$^*$-topology on $E^*$ for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weak$^*$-compact convex. Suppose the set $\Sigma = \{ y \in X : \sup_{y \in S(\hat{y})} [\inf_{w \in T(\hat{y})} Re\langle w, y - x \rangle + h(y) - h(x)] > 0 \}$ is open in $X$. Then there exists $\hat{y} \in X$ such that (i) $\hat{y} \in S(\hat{y})$ and (ii) there exists $\hat{w} \in T(\hat{y})$ with $Re\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Note that if $X$ is also bounded in Theorem 2.1, the map $S : X \to 2^X$ is, in addition, lower semicontinuous and for each $y \in \Sigma$, $T$ is upper semicontinuous at $y$ in $X$, then the set $\Sigma$ in Theorem 2.1 is always open in $X$ as can be seen in the proof of the following:
Theorem 2.3. Let $E$ be a locally convex Hausdorff topological vector space, $X$ be a non-empty paracompact convex and bounded subset of $E$ and $h : E \to \mathbb{R}$ be convex. Let $S : X \to 2^X$ be continuous such that each $S(x)$ is compact convex and $T : X \to 2^{E^*}$ be strongly $h$-pseudo-monotone and be upper semicontinuous from $co(A)$ to the weak*-topology on $E^*$ for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weak*-compact convex. Suppose that for each $y \in \Sigma = \{ y \in X : \sup_{x \in S(y) \setminus T(y)} \inf_{w \in T(y)} \Re \langle w, y - x \rangle + h(y) - h(x) > 0 \}$, $T$ is upper semicontinuous at $y$ from the relative topology on $X$ to the strong topology on $E^*$. Suppose further that there exist a non-empty compact subset $K$ of $X$ and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and

$$\inf_{w \in T(y)} \Re \langle w, y - x_0 \rangle + h(y) - h(x_0) > 0$$

for all $y \in X \setminus K$. Then there exists $\hat{y} \in K$ such that (i) $\hat{y} \in S(\hat{y})$ and (ii) there exists $\hat{w} \in T(\hat{y})$ with $\Re \langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Proof. By virtue of Theorem 2.1, we need only show that the set

$$\Sigma := \{ y \in X : \sup_{x \in S(y) \setminus T(y)} \inf_{w \in T(y)} \Re \langle w, y - x \rangle + h(y) - h(x) > 0 \}$$

is open in $X$. Indeed, let $y_0 \in \Sigma$; then there exists $x_0 \in S(y_0)$ such that $\alpha := \inf_{w \in T(y_0)} \Re \langle w, y_0 - x_0 \rangle + h(y_0) - h(x_0) > 0$.

Let $W := \{ w \in E^* : \sup_{z_1, z_2 \in X} |\langle w, z_1 - z_2 \rangle| < \frac{\alpha}{2} \}$. Then $W$ is a strongly open neighborhood of 0 in $E^*$ so that $U_1 := T(y_0) + W$ is an open neighborhood of $T(y_0)$ in $E^*$. Since $T$ is upper semicontinuous at $y_0$ in $X$, there exists an open neighborhood $N_1$ of $y_0$ in $X$ such that $T(y_0) \subset U_1$ for all $y \in N_1$.

Now, the rest of the proof is similar to the proof of Theorem 2.2 in [6]. Hence by the rest of the proof of Theorem 2.2 in [6], $\Sigma$ is open in $X$. This proves the theorem.

\[\square\]

If $X$ is compact, we obtain the following immediate consequence of Theorem 2.3:

Theorem 2.4. Let $E$ be a locally convex Hausdorff topological vector space, $X$ be a non-empty compact convex subset of $E$ and $h : E \to \mathbb{R}$ be convex. Let $S : X \to 2^X$ be continuous such that each $S(x)$ is closed convex and $T : X \to 2^{E^*}$ be strongly $h$-pseudo-monotone and be upper semicontinuous from $co(A)$ to the weak*-topology on $E^*$ for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weak*-compact convex. Suppose that for each $y \in \Sigma = \{ y \in X : \sup_{x \in S(y) \setminus T(y)} \inf_{w \in T(y)} \Re \langle w, y - x \rangle + h(y) - h(x) > 0 \}$, $T$ is upper semicontinuous at $y$ from the relative topology on $X$ to the strong topology on $E^*$. Then there exists $\hat{y} \in X$ such that (i) $\hat{y} \in S(\hat{y})$ and (ii) there exists $\hat{w} \in T(\hat{y})$ with $\Re \langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

We remark here that in Theorems 2.1-2.4, the condition “$h : E \to \mathbb{R}$ be convex” can be replaced by the condition “$h : X \to \mathbb{R}$ be convex such that $h|_{co(A)}$ is continuous for each $A \in \mathcal{F}(X)$”.

3. Generalized quasi-variational inequalities for pseudo-monotone operators

In this section we shall obtain some existence theorems of generalized quasi-variational inequalities for pseudo-monotone operators (Definition 1.1) on paracompact convex sets.
We shall first establish the following result:

**Theorem 3.1.** Let $E$ be a locally convex Hausdorff topological vector space, $X$ be a non-empty paracompact convex and bounded subset of $E$ and $h : E \to \mathbb{R}$ be convex such that $h(X)$ is bounded. Let $S : X \to 2^X$ be upper semicontinuous such that each $S(x)$ is compact convex and $T : X \to 2^{E^*}$ be $h$-pseudo-monotone and be upper semicontinuous from $co(A)$ to the weak*-topology on $E^*$ for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weak*-compact convex and $T(X)$ is strongly bounded. Suppose that the set $\Sigma = \{ y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} Re \langle w, y - x \rangle + h(y) - h(x)] > 0 \}$ is open in $X$. Suppose further that there exist a non-empty compact subset $K$ of $X$ and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and $\inf_{w \in T(y)} Re \langle w, y - x_0 \rangle + h(y) - h(x_0) > 0$ for all $y \in X \setminus K$. Then there exists $\bar{y} \in K$ such that (i) $\bar{y} \in S(\bar{y})$ and (ii) there exists $\hat{w} \in T(\bar{y})$ with $Re \langle \hat{w}, \bar{y} - x \rangle \leq h(x) - h(\bar{y})$ for all $x \in S(\bar{y})$.

**Proof.** We divide the proof into two steps:

**Step 1.** There exists a point $\hat{y} \in X$ such that $\hat{y} \in S(\bar{y})$ and

$$\sup_{x \in S(\bar{y})} \left[ \inf_{w \in T(\bar{y})} Re \langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x) \right] \leq 0.$$

Suppose the contrary. Then for each $y \in X$, either $y \not\in S(y)$ or there exists $x \in S(y)$ such that $\inf_{w \in T(y)} Re \langle w, y - x \rangle + h(y) - h(x) > 0$; that is, $y \not\in S(y)$ or $y \not\in \Sigma$. If $y \not\in S(y)$, then by Hahn-Banach separation theorem, there exists $p \in E^*$ such that $Re \langle p, y \rangle - \sup_{x \in S(y)} Re \langle p, x \rangle > 0$. For each $y \in X$, set $\gamma(y) := \sup_{x \in S(y)} [\inf_{w \in T(y)} Re \langle w, y - x \rangle + h(y) - h(x)]$. Let $V_0 := \{ y \in X : \gamma(y) > 0 \} = \Sigma$ and for each $p \in E^*$, set $V_p := \{ y \in X : Re \langle p, y \rangle - \sup_{x \in S(y)} Re \langle p, x \rangle > 0 \}$.

Then $X = V_0 \cup \bigcup_{p \in E^*} V_p$. Since each $V_p$ is open in $X$ by Lemma 1 in [11] and $V_0$ is open in $X$ by hypothesis, $\{V_0, V_p : p \in E^* \}$ is an open covering for $X$. Since $X$ is paracompact, there is a continuous partition of unity $\{\beta_0, \beta_p : p \in E^* \}$ for $X$ subordinated to the open cover $\{V_0, V_p : p \in E^* \}$. Note that for each $A \in \mathcal{F}(X)$, $h$ is continuous on $co(A)$ (see e.g. [10, Corollary 10.1.1, p.83]). Define $\phi : X \times X \to \mathbb{R}$ by

$$\phi(x, y) = \beta_0(y) \left[ \min_{w \in T(y)} Re \langle w, y - x \rangle + h(y) - h(x) \right] + \sum_{p \in E^*} \beta_p(y) Re \langle p, y - x \rangle$$

for each $x, y \in X$. Then we have the following.

1. The same argument in proving (1) in the proof of Theorem 2.1 shows that for each $A \in \mathcal{F}(X)$ and each fixed $x \in co(A)$, the map $y \mapsto \phi(x, y)$ is lower semicontinuous on $co(A)$.

2. The same argument in proving (2) in the proof of Theorem 2.1 shows that for each $A \in \mathcal{F}(X)$ and for each $y \in co(A)$, $\min_{x \in A} \phi(x, y) \leq 0$.

3. Suppose $A \in \mathcal{F}(X)$, $x, y \in co(A)$ and $\{y_\alpha\}_{\alpha \in \Gamma}$ is a net in $X$ converging to $y$ with $\phi(tx + (1 - t)y, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$ and all $t \in [0, 1]$.

**Case 1.** $\beta_0(y) = 0$.

Note that $\beta_0(y_\alpha) \geq 0$ for each $\alpha \in \Gamma$ and $\beta_0(y_\alpha) \to 0$. Since $T(X)$ is strongly bounded and $\{y_\alpha\}_{\alpha \in \Gamma}$ is a bounded net, it follows that

$$\limsup_{\alpha} [\beta_0(y_\alpha) \left( \min_{w \in T(y_\alpha)} Re \langle w, y_\alpha - x \rangle + h(y_\alpha) - h(x) \right)] = 0.$$
Also \( \beta_0(y) [\min_{w \in T(y)} \Re \langle w, y - x \rangle + h(y) - h(x)] = 0 \). Thus

\[
(3.2) \quad \limsup_{\alpha} [\beta_0(y_a)(\min_{w \in T(y_a)} \Re \langle w, y_a - x \rangle + h(y_a) - h(x))] + \sum_{p \in E^*} \beta_p(y) Re \langle p, y - x \rangle
\]

\[
= \sum_{p \in E^*} \beta_p(y) Re \langle p, y - x \rangle \quad \text{(by (3.1))}
\]

\[
= \beta_0(y)[\min_{w \in T(y)} \Re \langle w, y - x \rangle + h(y) - h(x)] + \sum_{p \in E^*} \beta_p(y) Re \langle p, y - x \rangle.
\]

For \( t = 1 \) we have \( \phi(x, y_a) \leq 0 \) for all \( \alpha \in \Gamma \), i.e.,

\[
(3.3) \quad \beta_0(y_a)[\min_{w \in T(y_a)} \Re \langle w, y_a - x \rangle + h(y_a) - h(x)] + \sum_{p \in E^*} \beta_p(y_a) Re \langle p, y_a - x \rangle \leq 0
\]

for all \( \alpha \in \Gamma \). Therefore

\[
\limsup_{\alpha} [\beta_0(y_a)(\min_{w \in T(y_a)} \Re \langle w, y_a - x \rangle + h(y_a) - h(x))] + \liminf_{\alpha} [\sum_{p \in E^*} \beta_p(y_a) Re \langle p, y_a - x \rangle]
\]

\[
\leq \limsup_{\alpha} [\beta_0(y_a)(\min_{w \in T(y_a)} \Re \langle w, y_a - x \rangle + h(y_a) - h(x))]
\]

\[
+ \sum_{p \in E^*} \beta_p(y_a) Re \langle p, y_a - x \rangle
\]

\[
\leq 0 \quad \text{(by (3.3)).}
\]

Thus

\[
(3.4) \quad \limsup_{\alpha} [\beta_0(y_a)(\min_{w \in T(y_a)} \Re \langle w, y_a - x \rangle + h(y_a) - h(x))] + \sum_{p \in E^*} \beta_p(y) Re \langle p, y - x \rangle \leq 0.
\]

Hence by (3.2) and (3.4), we have \( \phi(x, y) \leq 0 \).

Case 2. \( \beta_0(y) > 0 \).

Since \( \beta_0(y_a) \rightarrow \beta_0(y) \), there exists \( \lambda \in \Gamma \) such that \( \beta_0(y_a) > 0 \) for all \( \alpha \geq \lambda \).

Then for \( t = 0 \) we have \( \phi(y, y_a) \leq 0 \) for all \( \alpha \in \Gamma \), i.e.,

\[
\beta_0(y_a)[\min_{w \in T(y_a)} \Re \langle w, y_a - y \rangle + h(y_a) - h(y)] + \sum_{p \in E^*} \beta_p(y_a) Re \langle p, y_a - y \rangle \leq 0
\]

for all \( \alpha \in \Gamma \). Thus

\[
(3.5) \quad \limsup_{\alpha} [\beta_0(y_a)(\min_{w \in T(y_a)} \Re \langle w, y_a - y \rangle + h(y_a) - h(y))] + \sum_{p \in E^*} \beta_p(y_a) Re \langle p, y_a - y \rangle] \leq 0.
\]
Since \( \beta \geq 0 \), \( \beta_0(y_\alpha) \) is \( h \)-pseudo-monotone, we have
\[
\sum_{p \in E^*} \beta_p(y_\alpha) \Re \langle p, y_\alpha - y \rangle = 0 \quad \text{by (3.5)}. 
\]

Since \( \liminf_{\alpha} [\sum_{p \in E^*} \beta_p(y_\alpha) \Re \langle p, y_\alpha - y \rangle] = 0 \), we have
\[
\limsup_{\alpha} [\beta_0(y_\alpha) \Re \langle w, y_\alpha - y \rangle] = 0. 
\]

Since \( \beta_0(y_\alpha) > 0 \) for all \( \alpha \geq \lambda \), it follows that
\[
\beta_0(y) \limsup_{\alpha} \left[ \min_{w \in T(y_\alpha)} \Re \langle w, y_\alpha - y \rangle + h(y_\alpha) - h(y) \right] = \limsup_{\alpha} [\beta_0(y_\alpha) \Re \langle w, y_\alpha - y \rangle + h(y_\alpha) - h(y)] \geq 0. 
\]

Since \( \beta_0(y) > 0 \), by (3.6) and (3.7) we have
\[
\limsup_{\alpha} \left[ \min_{w \in T(y_\alpha)} \Re \langle w, y_\alpha - y \rangle + h(y_\alpha) - h(y) \right] = 0. 
\]

Since \( T \) is \( h \)-pseudo-monotone, we have
\[
\liminf_{\alpha} \left[ \min_{w \in T(y_\alpha)} \Re \langle w, y_\alpha - x \rangle + h(y_\alpha) - h(x) \right] \geq \min_{w \in T(y)} \Re \langle w, y - x \rangle + h(y) - h(x). 
\]

Since \( \beta_0(y) > 0 \), we have
\[
\beta_0(y) \left[ \liminf_{\alpha} \left[ \min_{w \in T(y_\alpha)} \Re \langle w, y_\alpha - x \rangle + h(y_\alpha) - h(x) \right] \right] \geq \beta_0(y) \left[ \min_{w \in T(y)} \Re \langle w, y - x \rangle + h(y) - h(x) \right]. 
\]

Thus
\[
\beta_0(y) \left[ \liminf_{\alpha} \left[ \min_{w \in T(y_\alpha)} \Re \langle w, y_\alpha - x \rangle + h(y_\alpha) - h(x) \right] \right] + \sum_{p \in E^*} \beta_p(y) \Re \langle p, y - x \rangle \geq \beta_0(y) \left[ \min_{w \in T(y)} \Re \langle w, y - x \rangle + h(y) - h(x) \right] + \sum_{p \in E^*} \beta_p(y) \Re \langle p, y - x \rangle. 
\]

For \( t = 1 \) we also have \( \phi(x, y_\alpha) \leq 0 \) for all \( \alpha \in \Gamma \), i.e.,
\[
\beta_0(y_\alpha) \left[ \min_{w \in T(y_\alpha)} \Re \langle w, y_\alpha - x \rangle + h(y_\alpha) - h(x) \right] + \sum_{p \in E^*} \beta_p(y_\alpha) \Re \langle p, y_\alpha - x \rangle \leq 0 
\]
for all \( \alpha \in \Gamma \). Therefore
\[
0 \geq \liminf_{\alpha} \left[ \beta_0(y_\alpha) \left( \min_{w \in T(y_\alpha)} \Re \langle w, y_\alpha - x \rangle + h(y_\alpha) - h(x) \right) \right.
\quad + \sum_{p \in E^*} \beta_p(y_\alpha) \Re \langle p, y_\alpha - x \rangle \bigg] \\
\geq \liminf_{\alpha} \left[ \beta_0(y_\alpha) \left( \min_{w \in T(y_\alpha)} \Re \langle w, y_\alpha - x \rangle + h(y_\alpha) - h(x) \right) \right.
\quad + \liminf_{\alpha} \left[ \sum_{p \in E^*} \beta_p(y_\alpha) \Re \langle p, y_\alpha - x \rangle \bigg] \\
= \beta_0(y) \liminf_{\alpha} \left[ \min_{w \in T(y_\alpha)} \Re \langle w, y_\alpha - x \rangle + h(y_\alpha) - h(x) \right] \\
\quad + \sum_{p \in E^*} \beta_p(y) \Re \langle p, y - x \rangle.
\]
(3.9)

Consequently, by (3.8) and (3.9), we have \( \phi(x, y) \leq 0 \).

Now, the rest of the proof of Step 1 is similar to the proofs in Step 1 of Theorem 2.1 and Theorem 3.1 in [6]. Thus Step 1 is proved.

**Step 2.** There exists a point \( \hat{w} \in T(\hat{y}) \) such that \( \Re \langle \hat{w}, \hat{y} - x \rangle + h(\hat{y}) - h(x) \leq 0 \) for all \( x \in S(\hat{y}) \).

Also the same proof of Step 2 of Theorem 2.1 shows that there exists \( \hat{w} \in T(\hat{y}) \) such that \( \Re \langle \hat{w}, \hat{y} - x \rangle + h(\hat{y}) - h(x) \leq 0 \) for all \( x \in S(\hat{y}) \).

If \( X \) is compact, we obtain the following immediate consequence of Theorem 3.1:

**Theorem 3.2.** Let \( E \) be a locally convex Hausdorff topological vector space, \( X \) be a non-empty compact convex subset of \( E \) and \( h : E \to \mathbb{R} \) be convex such that \( h(X) \) is bounded. Let \( S : X \to 2^X \) be upper semicontinuous such that each \( S(x) \) is closed convex and \( T : X \to 2^{E^*} \) be h-pseudo-monotone and be upper semicontinuous from \( \text{co}(A) \) to the weak*-topology on \( E^* \) for each \( A \in \mathcal{F}(X) \) such that each \( T(x) \) is weak*-compact convex and \( T(X) \) is strongly bounded. Suppose that the set \( \Sigma = \{ y \in X : \sup_{x \in S(y)} \left[ \inf_{w \in T(y)} \Re \langle w, y - x \rangle + h(y) - h(x) \right] > 0 \} \) is open in \( X \). Then there exists \( \hat{y} \in X \) such that (i) \( \hat{y} \in S(\hat{y}) \) and (ii) there exists \( \hat{w} \in T(\hat{y}) \) with \( \Re \langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \) for all \( x \in S(\hat{y}) \).

Note that if the map \( S : X \to 2^X \) is, in addition, lower semicontinuous and for each \( y \in \Sigma \), \( T \) is upper semicontinuous at \( y \) in \( X \), then the set \( \Sigma \) in Theorem 3.1 is always open in \( X \) as can be seen in the proof of the following:

**Theorem 3.3.** Let \( E \) be a locally convex Hausdorff topological vector space, \( X \) be a non-empty paracompact convex and bounded subset of \( E \) and \( h : E \to \mathbb{R} \) be convex such that \( h(X) \) is bounded. Let \( S : X \to 2^X \) be continuous such that each \( S(x) \) is compact convex and \( T : X \to 2^{E^*} \) be h-pseudo-monotone and be upper semicontinuous from \( \text{co}(A) \) to the weak*-topology on \( E^* \) for each \( A \in \mathcal{F}(X) \) such that each \( T(x) \) is weak*-compact convex and \( T(X) \) is strongly bounded. Suppose that for each \( y \in \Sigma = \{ y \in X : \sup_{x \in S(y)} \left[ \inf_{w \in T(y)} \Re \langle w, y - x \rangle + h(y) - h(x) \right] > 0 \} \), \( T \) is upper semicontinuous at \( y \) from the relative topology on \( X \) to the strong topology on \( E^* \). Suppose further that there exist a non-empty compact subset \( K \) of \( X \) and a point \( x_0 \in X \) such that \( x_0 \in K \cap S(y) \) and \( \inf_{w \in T(y)} \Re \langle w, y - x_0 \rangle + h(y) - h(x_0) > 0 \) for all \( y \in X \setminus K \). Then there exists \( \hat{y} \in K \) such that (i) \( \hat{y} \in S(\hat{y}) \) and (ii) there exists \( \hat{w} \in T(\hat{y}) \) with \( \Re \langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \) for all \( x \in S(\hat{y}) \).
Proof. By virtue of Theorem 3.1, we need only show that the set \( \Sigma := \{ y \in X : \sup_{x \in S(y)} \left[ \inf_{w \in T(y)} R \langle w, y - x \rangle + h(y) - h(x) \right] > 0 \} \) is open in \( X \).

Now, following the same arguments as in the proofs of Theorems 3.2 in [6] and Theorem 2.3, we can similarly show that the set \( \Sigma \) is open in \( X \). Hence by Theorem 3.1 the conclusion follows.

If \( X \) is compact, we obtain the following immediate consequence of Theorem 3.3:

**Theorem 3.4.** Let \( E \) be a locally convex Hausdorff topological vector space, \( X \) be a non-empty compact convex subset of \( E \) and \( h : E \to \mathbb{R} \) be convex such that \( h(X) \) is bounded. Let \( S : X \to 2^X \) be continuous such that each \( S(x) \) is closed convex and \( T : X \to 2^{E^*} \) be \( h \)-pseudo-monotone and be upper semicontinuous from \( \text{co}(A) \) to the weak* topology on \( E^* \) for each \( A \in F(X) \) such that each \( T(x) \) is weak*-compact convex and \( T(X) \) is strongly bounded. Suppose that for each \( y \in \Sigma = \{ y \in X : \sup_{x \in S(y)} \left[ \inf_{w \in T(y)} R \langle w, y - x \rangle + h(y) - h(x) \right] > 0 \} \), \( T \) is upper semicontinuous at \( y \) from the relative topology on \( X \) to the strong topology on \( E^* \). Then there exists \( \tilde{y} \in X \) such that (i) \( \tilde{y} \in S(\tilde{y}) \) and (ii) there exists \( \tilde{w} \in T(\tilde{y}) \) with \( R \langle \tilde{w}, \tilde{y} - x \rangle \leq h(x) - h(\tilde{y}) \) for all \( x \in S(\tilde{y}) \).

We remark here that in Theorems 3.1-3.4, the condition “\( h : E \to \mathbb{R} \) be convex” can be replaced by the condition “\( h : X \to \mathbb{R} \) be convex such that \( h|_{\text{co}(A)} \) is continuous for each \( A \in F(X) \)”.

**References**


Department of Mathematics, Statistics and Computing Science, Dalhousie University, Halifax, Nova Scotia, Canada B3H 3J5

E-mail address: mohammad@mscs.dal.ca

E-mail address: kktan@mscs.dal.ca