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ON ANTOSIK'S LEMMA AND THE ANTOSIK-MIKUSINSKI BASIC MATRIX THEOREM

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Abstract. That Antosik's Lemma is not a special case of the Antosik-Mikusinski Basic Matrix Theorem will be shown and, an equivalent form of the Antosik-Mikusinski Basic Matrix Theorem will also be presented in this paper.

In [1] and [2], Antosik proved two results which are called the Antosik-Mikusinski Basic Matrix Theorem and Antosik's Lemma, respectively. The theorem and lemma have been proven to be quite effective in treating various topics in Functional Analysis and Set Function Theory [1]–[6]. In [6, 2.2], Swartz thought that Antosik's Lemma is a special case of the Antosik-Mikusinski Basic Matrix Theorem. In [7], Li Ronglu presented the Uniform Convergent Principle. Now, we will show that Swartz's conclusion is incorrect and, the Uniform Convergent Principle is an equivalent form of the Antosik-Mikusinski Basic Matrix Theorem.

Lemma 1 (Antosik)**.** *Let G be an abelian topological group and $x_{ij} \in G$ for $i, j \in \mathbf{N}$. Suppose that each strictly increasing sequence $\{m_i\}$ in $\mathbf{N}$ has a subsequence $\{n_i\}$ such that*
(i) $\lim_i x_{n_i n_j} = 0$ *for all* $j \in \mathbf{N}$, *and*
(ii) $\lim_i \sum_{j=1}^{\infty} x_{n_i n_j} = 0$.
Then $\lim x_{ii} = 0$.

Antosik observed that assumption (i) can be dropped if G is a locally convex space and posed the problem ([2]) of whether (i) can also be dropped in general. In [8], Weber solved this problem showing that assumption (i) is in fact superfluous.

A direct consequence of Lemma 1 is as follows:

Corollary 1. *Let G be an abelian topological group and $z_{ij} \in G$ for $i, j \in \mathbf{N}$. Suppose that*
(I) $\lim_i z_{ij} = 0$ *for each* $j \in \mathbf{N}$, *and*
(II) *for each strictly increasing sequence of positive integers $\{m_j\}$ in $\mathbf{N}$ there is a subsequence $\{n_j\}$ such that* $\lim_i \sum_{j=1}^{\infty} z_{in_j} = 0$.
Then $\lim_i z_{ii} = 0$.

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Now, we show that the following Antosik-Mikusinski Basic Matrix Theorem can be obtained from Corollary 1.

Theorem 1 (Antosik-Mikusinski)**.** *Let G be an abelian topological group, $x_{ij} \in G$ for $i, j \in \mathbf{N}$. Suppose*

(III) $\lim_i x_{ij} = x_j$ *exists for each j, and*

(IV) *for each strictly increasing sequence of positive integers $\{m_j\}$ there is a subsequence $\{n_j\}$ such that $\{\sum_j x_{in_j}\}_{i=1}^{\infty}$ is a Cauchy sequence.*

Then $\lim_i x_{ij} = x_j$ uniformly for $j \in \mathbf{N}$. In particular, $\lim_i x_{ii} = 0$.

Proof. If the conclusion fails, then there exist a closed, symmetric neighbourhood V_0 of 0 in G and strictly increasing sequences of positive integers $\{p_k\}$ and $\{q_k\}$ such that

$$x_{p_k q_k} - x_{q_k} \notin V_0 \tag{1}$$

for all k. Pick a closed, symmetric neighbourhood V_1 of 0 such that $V_1 + V_1 \subseteq V_0$. Note that $x_{p_i q_j} - x_{q_j} \to 0$ as $i \to \infty$ for $j \in \mathbf{N}$. Therefore, there exists a subsequence $\{m_i\}$ of $\{p_i\}$ such that

$$x_{m_i q_i} - x_{q_i} \in V_1 \tag{2}$$

for $i \in \mathbf{N}$. We have

$$x_{p_i q_i} - x_{q_i} = (x_{p_i q_i} - x_{m_i q_i}) + (x_{m_i q_i} - x_{q_i}). \tag{3}$$

Consider the matrix $(x_{p_i q_j} - x_{m_i q_j})$ and note that the matrix satisfies conditions of Corollary 1. Consequently,

$$x_{p_i q_i} - x_{m_i q_i} \to 0$$

as $i \to \infty$, and

$$x_{p_i q_i} - x_{m_i q_i} \in V_1$$

for sufficiently large i. Hence, by (3) and (2)

$$x_{p_i q_i} - x_{q_i} \in V_1 + V_1 \subseteq V_0$$

for sufficiently large i. Which contradicts (1) and we established the result.

Thus, we have Lemma 1 $\Rightarrow$ Corollary 1 $\Longleftrightarrow$ Theorem 1. Now, we show that Lemma 1 is not a special case of Theorem 1. □

Example 1. Consider the matrix (x_{ij}) such that $x_{ij} = 0$ if $i \neq j+1$ and $x_{ii-1} = 1$. Then (x_{ij}) satisfies the assumptions of Lemma 1. If Lemma 1 was a special case of Theorem 1, then the columns should converge to 0 uniformly. But they do not converge to 0 uniformly.

In this way, we have corrected the incorrect statement in ([6, 2.2]).

Now, we use Theorem 1 to prove Theorem 2 below.

Theorem 2 (Uniform Convergent Principle)**.** *Let G be an abelian topological group and let Ω be a sequentially compact topological space. Let $\{f_i\}$ be a sequence of sequentially continuous G-valued functions defined on Ω. If each strictly increasing sequence $\{m_j\}$ in $\mathbf{N}$ has a subsequence $\{n_j\}$ such that for each $\omega \in \Omega$, the series $\sum_j f_{n_j}(\omega)$ converges and $\sum_j f_{n_j} : \Omega \to G$ is sequentially continuous, then $\lim_j f_j(\omega) = 0$ uniformly with respect to $\omega \in \Omega$.*

Proof. We will show that $f_j(\omega) \longrightarrow 0$ uniformly for ω in Ω or, equivalently, for each sequence $\{\omega_i\}$ in Ω,

$$f_i(\omega_i) \longrightarrow 0 \tag{4}$$

as $i \longrightarrow \infty$. Let $\{\omega_i\}$ be a sequence of Ω. Since Ω is sequentially compact, there exists a subsequence $\{\omega_{n_i}\}$ of $\{\omega_i\}$ and ω_0 such that $\omega_{n_i} \longrightarrow \omega_0$. Consider the matrix $(f_{n_j}(\omega_{n_i}) - f_{n_j}(\omega_0))$. Note that the matrix satisfies conditions of Theorem 1. Therefore,

$$f_{n_i}(\omega_{n_i}) - f_{n_i}(\omega_0) \longrightarrow 0.$$

Since $f_{n_i}(\omega_0) \longrightarrow 0$, we get $f_{n_i}(\omega_{n_i}) \longrightarrow 0$. Hence, by assumption conditions, (4) holds. Thus, Theorem 1 $\Longrightarrow$ Theorem 2. □

Theorem 3. *Theorem 2 is an equivalent form of Theorem 1.*

Proof. Since Corollary 1 $\Longleftrightarrow$ Theorem 1 $\Longrightarrow$ Theorem 2, we only need to show that Theorem 2 $\Longrightarrow$ Corollary 1.

Indeed, assume the conditions of Corollary 1 are satisfied. Let $\Omega = \{\frac{1}{i}, 0\}_{i=1}^{\infty}$, for $x, y \in \Omega$, put $d(x,y) = |x-y|$. Then (Ω, d) is a sequentially compact topological space. Let $f_j : \Omega \longrightarrow G$ satisfy that if $\omega = \frac{1}{i}, f_j(\omega) = z_{ij}$; if $\omega = 0, f_j(\omega) = 0$. It is easily shown that each f_j is continuous, and for each strictly increasing sequence $\{m_j\}$ in $\mathbf{N}$ has subsequence $\{n_j\}$ such that for $i \in \mathbf{N}$, the series $\sum_j z_{in_j}$ is convergent and $\lim_i \sum_j z_{in_j} = 0$. Thus, for each $\omega \in \Omega$, the series $\sum_j f_{n_j}(\omega)$ is convergent and $\sum_j f_{n_j} : \Omega \longrightarrow G$ is continuous. It follows from Theorem 2 that $\lim_j f_j(\omega) = 0$ uniformly with respect to $\omega \in \Omega$. In particular, $\lim_j f_j(\frac{1}{j}) = \lim_j z_{jj} = 0$. So Corollary 1 is proved. □

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