ALGEBRAIC GROUPS OVER FINITE FIELDS, 
A QUICK PROOF OF LANG’S THEOREM

PETER MÜLLER

(Communicated by Stephen D. Smith)

Abstract. We give an easy proof of Lang’s theorem about the surjectivity of the Lang map $g \mapsto g^{-1}F(g)$ on a linear algebraic group defined over a finite field, where $F$ is a Frobenius endomorphism.

Let $G \leq \text{GL}_n(k)$ be a connected linear algebraic group over an algebraic closure $k$ of a finite field. For $q$ a power of the characteristic of $k$, the endomorphism $[q]$ of $\text{GL}_n(k)$ which raises the entries of $\text{GL}_n(k)$ to the $q$th power is called the standard Frobenius endomorphism of $\text{GL}_n(k)$. An endomorphism $F$ of $G$ (as algebraic groups) is called a Frobenius endomorphism of $G$ if some power of $F$ is the restriction of $[q]$ to $G$. The important theorem of Lang [4], in the version used in the theory of finite groups of Lie type, is

**Theorem.** The map $g \mapsto g^{-1}F(g)$ is surjective on $G$.

The usual argument uses differentials; see [1, 16.4], [2, 3.10], [5, 3.3.16]. In [6], Steinberg gives a different argument avoiding differentials, showing that $g \mapsto g^{-1}F(g)$ is a finite morphism of $G$ to itself.

We give yet another proof, which rests on very basic (and easily proven) properties of algebraic groups. The argument can be used at an early stage in textbooks on algebraic groups.

**Proof.** The group $G$ acts morphically from the right on itself, where $g \in G$ sends $x \in G$ to $g^{-1}xF(g)$. By [3, 8.3], there is a closed orbit $\Omega$. Choose $x \in \Omega$. Lang’s theorem follows from connectivity of $G$ once we know that $\Omega$ has the same dimension as $G$, because then $G = \Omega$, so $G$ is also the orbit through 1. By [3, 4.1], we need to show that $g^{-1}xF(g) = x$ has only finitely many solutions $g \in G$. Rewrite this as $f(g) = g$, where $f(g) := xF(g)x^{-1}$. Let $m$ be a positive integer such that $F^m$ is a standard Frobenius endomorphism with $x = F^m(x)$. Let $r$ be the order of the group element $xF(x)F^2(x)\cdots F^{m-1}(x)$. Then $F^{mr}(g) = F^{mr}(g)$ for all $g \in G$. So $F^{mr}(g) = g$ has only finitely many solutions in $G$, so this is even more true for the equation $f(g) = g$. The claim follows.

**Remark.** It would be interesting to prove the more general Lang-Steinberg theorem in a similar fashion. This theorem (see [4]) states the following: Let $\sigma$ be an endomorphism of the connected linear algebraic group $G$ over any algebraically

Received by the editors August 23, 2001 and, in revised form, September 26, 2001.
2000 Mathematics Subject Classification. Primary 20G40.
closed field such that $\sigma$ fixes only finitely many elements of $G$. Then $g \mapsto g^{-1}\sigma(g)$ is surjective on $G$.

References


IWR, Universität Heidelberg, Im Neuenheimer Feld 368, 69120 Heidelberg, Germany

E-mail address: Peter.Mueller@iwr.uni-heidelberg.de